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Théorèmes limites en géométrie stochastique et marginales de mesures aléatoires

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Je remercie de tout mon coeur ma compagne Séverine, pour son soutien, et ma fille Romane, à l'influence grandissante, et à qui je dédie ce mémoire.

Note au lecteur

Ce manuscrit propose un regroupement de mes activités de recherche, en développant plus particulièrement deux axes : les théorèmes limites en géométrie aléatoire (Première partie), et l'étude des marginales de processus ponctuels ou d'ensembles aléatoires (Seconde partie). Pour chaque question abordée, je présente les contributions apportées par mes co-auteurs et moi-même, des éléments de contexte, ainsi que les principaux travaux ayant participé à la même ligne de recherche.

Tous les travaux présentés ont été publiés dans des revues avec comité de lecture, pour cette raison la plupart des preuves ne sont pas reproduites ici, afin de proposer une lecture fluide. Quand c'est pertinent, on donne des éléments expliquant le déroulement de la preuve, et ses points les plus novateurs. Pour harmoniser les notations du présent manuscrit, les notations originales ont été modifiées dans un souci de cohérence. Pour certains travaux, seuls les préambules sont présentés, ce qui est indiqué par une référence explicite à l'article dans le titre de la section.

Introduction

L'objet central de mon travail, et d'une grande partie de la géométrie stochastique, est l'étude de grands système spatiaux désordonnés et homogènes. Donnons quelques exemples de tels systèmes pour aiguiller l'intuition :

- Un ensemble dénombrable et aléatoire de points isolés pouvant représenter les particules d'un gaz, les étoiles de l'univers, ou les antennes d'un réseau de télécommunications. On appelle *processus ponctuel* ce type d'objet, voir les figures 3 et 4. Les processus ponctuels poissoniens en sont un exemple éminemment étudié, et sont un objet majeur en géométrie stochastique et dans mes travaux, voir toute la **Partie I** et le Chapitre 8.

- Un ensemble aléatoire constitué de deux composantes, aussi appelées *phases* , que l'on peut imaginer coloriées en noir et blanc. Le blanc peut par exemple représenter la partie vide au sein d'une mousse, ou d'une éponge, ou un des fluides dans un mélange de deux fluides non miscibles, voir la figure 1, et les chapitres 6 et 9.

- Dans l'exemple précédent, le blanc peut aussi représenter les terres émergées dans une partie du globe (et le noir l'océan). Cet ensemble aléatoire s'obtient en seuillant la fonction *altitude* au niveau 0, c'est un *ensemble de niveau supérieur* de cette fonction, ou une *excursion* multi-dimensionnelle, voir la figure 5. Cette fonction peut être modélisée par un champ aléatoire sur $\mathbb{R}^2$ ou sur la sphère unité de $\mathbb{R}^3$. Les excursions aléatoires interviennent dans tout problème impliquant des données récoltées après seuillage d'un champ au-dessus ou au-dessous d'une certaine valeur, on les retrouve en cosmologie dans l'étude du champ diffus cosmologique, en neuro-biologie dans l'étude de l'activité électrique du cerveau, et dans bien d'autres domaines. Voir les chapitres 4.1 et 7.

- Une mosaïque, c'est-à-dire une partition du plan par des ensembles bornés, les *cellules*, habituellement convexes, ou de manière duale le réseau de segments qui délimitent ces cellules, voir la figure 2. Ces objets peuvent modéliser des réseaux d'infrastructure, ou l'organisation de cellules dans un tissu. En dimension $d \geqslant 3$, le réseau dual est constitué de polygones de dimension $d-1$. Voir la figure 2, et les chapitres 4.4 et 10.

Le caractère *homogène* est modélisé par la propriété de *stationarité*, c'est-à-dire l'invariance de la loi du modèle sous l'effet d'un groupe de transformations. Un paradigme très fréquent est d'observer le modèle sur une grande fenêtre, et le comportement asymptotiques de certaines observables lorsque la fenêtre tend vers l'espace tout entier. Parfois c'est la fenêtre qui est fixée et l'ensemble observé devient de plus en plus dense, voir la figure 3 et l'exemples des enveloppes convexes aléatoires. Les exemples de statistiques les plus immédiatement calculable sont le nombre de points, la mesure de la surface couverte,

FIGURE 1 – Excursions au niveau 0 de plusieurs champs aléatoires gaussiens présentant différents niveaux de régularité [Lan02]

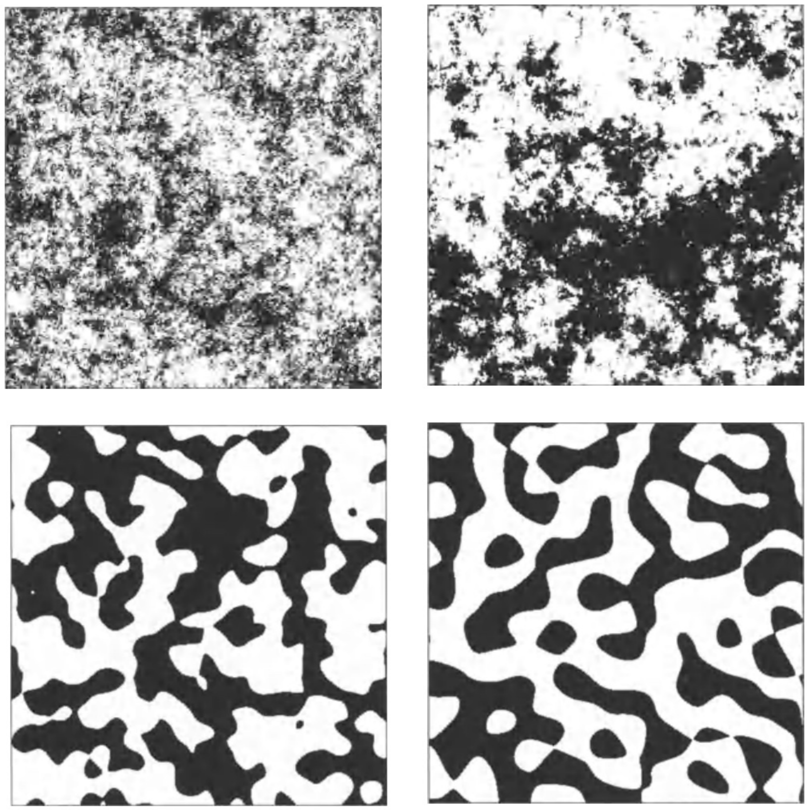
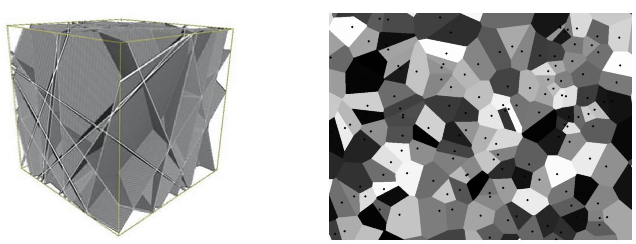


FIGURE 2 – *Gauche :* une mosaïque poissonienne d'hyperplans en trois dimensions. *Droite :* Mosaïque de Poisson-Voronoï en dimension 2



ou la longueur totale des arêtes visibles, mais on peut largement complexifier ces observables : nombre de composantes connexes bornées, nombre d'amas de trois points dans une boule de volume 1, etc... J'ai décomposé mes travaux en deux axes de recherche.

Première partie : Théorèmes limites en géométrie aléatoire

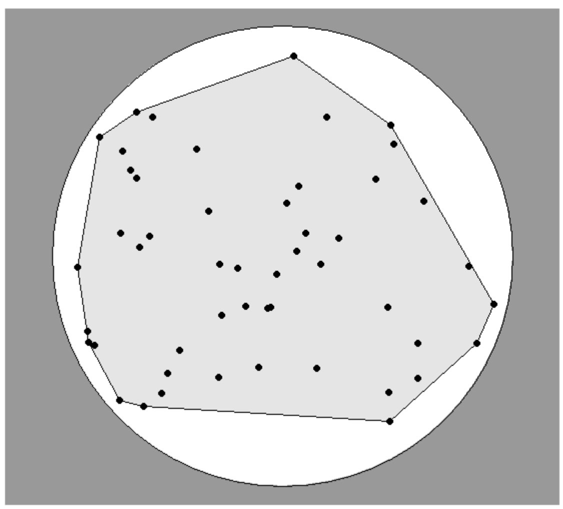
On s'attache à définir, pour certains modèles et certaines fonctionnelles géométriques comme ci-dessus, des conditions sous lesquelles on a un comportement asymptotique normal, après renormalisation par la variance (théorème central limite). En particulier, on tente de donner une vitesse de convergence optimale vers la loi gaussienne pour une distance donnée, souvent pour la distance de Kolmogorov, qui est adaptée à l'établissement d'intervalles de confiance.

Commençons par donner un exemple d'un tel résultat, obtenu dans [LSY19], qui a permis d'améliorer l'état de l'art. Soit $\xi_n = \{X_1, \ldots, X_n\}$ des points indépendants et uniformément distribués dans la boule unité $B(0,1)$ de $\mathbb{R}^d$, et $\mathcal{C}(\xi_n)$ l'enveloppe convexe de ξ_n (voir la figure 3). On considère une fonctionnelle $F(\mathcal{C}(\xi_n)) =: G_n$ de nature additive, par exemple :

- nombre de facettes de $\mathcal{C}(\xi_n)$,
- volume $\mathcal{L}^d(\mathcal{C}(\xi_n))$, où $\mathcal{L}^d$ est la mesure de Lebesgue d-dimensionnelle,
- surface $\mathcal{H}^{d-1}(\partial\mathcal{C}(\xi_n))$, où $\mathcal{H}^{d-1}$ est la mesure de Hausdorff,

voir [LSY19] pour la description des fonctionnelles additives admissibles.

Figure 3 – Enveloppe convexe de 50 points indépendants tirés uniformément dans la boule unité



Il a été montré dans Reitzner [Rei05] qu'il existe $c_+ > c_- > 0$ telles que la variance de G_n vérifie

$$c_- < \frac{\mathrm{Var}(G_n)}{n^{\frac{d-1}{d+1}}} < c_+$$

pour n suffisamment grand. Reitzner a également montré le théorème central limite

$$\tilde{G}_n := \frac{G_n - \mathbf{E}(G_n)}{\sqrt{\mathrm{Var}(G_n)}} \xrightarrow[n\to\infty]{\text{(law)}} N,$$

où N est une variable gaussienne standard. En utilisant le résultat établi conjointement avec Peccati [LP17], nous avons établi avec Shulte et Yukich [LSY19] la vitesse de convergence suivante : il existe $C > 0$ tel que

$$\sup_{t\in\mathbb{R}} \left|\mathbf{P}(\tilde{G}_n \geqslant t) - \mathbf{P}(N \geqslant t)\right| \leqslant Cn^{-\frac{d-1}{2(d+1)}}, n \in \mathbb{N}.$$

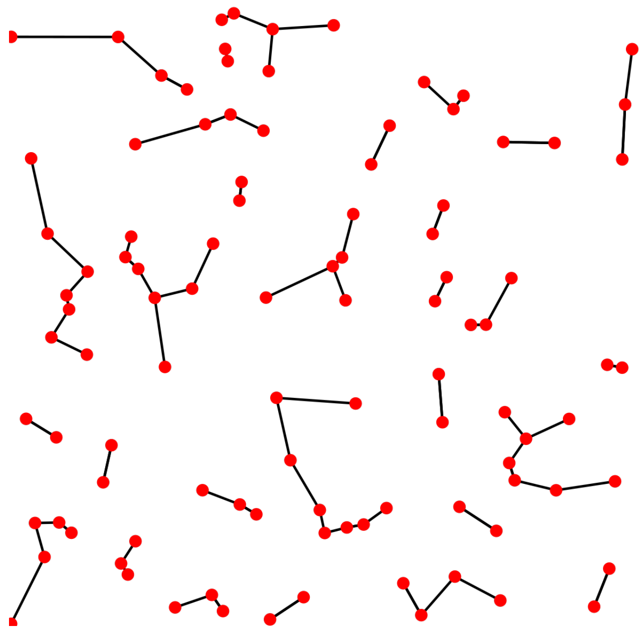
Ces ordres de gandeur de variance et de vitesse de convergence sont habituels pour des fonctionnelles de type *surface order scaling*, c'est-à-dire lorsque la valeur de G_n est essentiellement déterminée par les X_i qui sont proches de la frontière d'un ensemble déterministe de frontière lisse, ici la boule unité. Autrement dit, si l'on efface les points qui tombent à une distance $n^{-1/d}\log(n)$ de la sphère unité, il y a peu de chances que la valeur de G_n change. On donne dans [LSY19] d'autres exemples de fonctionnelles de ce type, comme l'*approximation Voronoi d'un ensemble*, ou l'étude des *points maximaux relatifs à un cône*. Nous retrouvons également le même ordre de grandeur pour la vitesse de convergence vers une loi gaussienne, ce qui laisse à penser que cet ordre de grandeur est le meilleur possible, même si l'optimalité est difficile à montrer dans ce genre de problème.

Nous donnons en fait dans [LSY19] un résultat plus général valide pour les fonctionnelles dite *stabilisantes*, c'est-à-dire où chaque X_i apporte une contribution qui ne dépend que des autres points aléatoires avoisinant. Autrement dit, la contribution d'un point X_i ne change pas si l'on efface les points aléatoires qui tombent à une distance de X_i supérieure à $n^{-1/d}\log(n)$. Dans l'exemple du nombre de sommets de l'enveloppe convexe, la contribution d'un point est de 1 s'il est extrêmal, et 0 sinon, et le caractère extrêmal d'un point ne dépend effectivement que des points alentour. Le principe de stabilisation a une portée bien plus grande que les fonctionnelles de type *surface order scaling*. Un exemple classique également développé dans [LSY19] est la longueur $L(\xi_n)$ du graphe des plus proche voisins basé sur ξ_n, construit en reliant par une arête chaque point X_i au point X_j le plus proche, avec $j \neq i$ (voir la figure 4, où cette fois les points X_i ont été tirés uniformément dans un carré). Cette fonctionnelle est de type *volume order scaling*, c'est-à-dire que, hormis des effets de bord négligeables, les points donnent tous des contributions du même ordre. Autrement dit on ne peut présumer la contribution du point X_i en se basant uniquement sur son emplacement, contrairement au cas de l'enveloppe convexe. Ces graphes ont trouvé de nombreuses applications, notamment récemment dans le cadre de l'analyse de données en grandes dimension [LB05], et on donne dans [LSY19] des vitesses de convergence vraisemblablement optimales vers la loi gaussienne (après renormalisation), pour de nombreuses statistiques basées sur les graphes des plus proches voisins.

Donnons désormais un bref historique récent du sujet. Giovanni Peccati, Ivan Nourdin, David Nualart et leurs co-auteurs ont développé une approche combinant la *méthode de Stein* et le calcul de Malliavin pour obtenir des théorèmes limites, et en particulier des résultats de normalité asymptotique, pour des variables aléatoires dont l'aléa vient d'un processus gaussien [NP12]. En utilisant une approche similaire, Peccati, Solé, Utzet et Taqqu [Pec+10], dans le cadre ou l'aléa est une mesure poissonienne, ont écrit une inégalité générale du type *Berry-Esseen*, c'est-à-dire donnant la distance entre une fonctionnelle poissonienne et une variable gaussienne, en vue de quantifier le théorème de la limite centrale dans ce contexte. Cette inégalité exploite une décomposition orthogonale dont jouissent toutes les variables de carré intégrable mesurables par rapport à un processus poissonien : la *décomposition de Wiener-Itô*, et fait appel aux opérateurs de Malliavin que l'on peut définir sur l'espace de Hilbert induit par cette décomposition.

Concomitamment à mon arrivée en post-doctorat à l'Université de Luxembourg en 2011, Matthias Schulte, et son directeur de thèse Matthias Reitzner, ont exploité le résultat de [Pec+10], et les travaux de Last et Penrose [LP11] sur la décomposition de Wiener-Itô, pour montrer des théorèmes centraux limites pour un certain type de fonctionnelles géométriques ayant une décomposition finie : les U-statistiques [RS13]. Ce travail a été le point de départ de nos travaux avec Giovanni Peccati [LP13a ; LP13b], ainsi que de

FIGURE 4 – Graphe des plus proches voisins de 100 points indépendants tirés uniformément dans $[0,1]^2$.



nombreux autres travaux sur les théorèmes limites utilisant la méthode Stein-Malliavin en géométrie stochastique. Nous avons étudié des U-statistiques géométriques à différentes échelles, en utilisant notamment les contractions des noyaux de leur décomposition, et avons appliqué ces résultats à l'étude générale de graphes géométriques. Nous avons utilisé cette approche pour établir un *4th moment theorem*, qui montre que sous certaines conditions, une combinaison linéaire de variables ayant une décomposition finie converge vers une variables gaussienne dès lors que ses 4 premiers moments convergent vers ceux de la variable gaussienne. Ces travaux se limitent aux variables ayant une décomposition finie car l'inégalité fait apparaître des constantes difficiles à contrôler pour les termes d'ordre supérieur.

Par la suite, Last, Peccati et Schulte [LPS16] ont développé une autre inégalité de type Berry-Esseen, mais cette fois ne faisant apparaître que les dérivées de Malliavin d'ordre 1 et 2, et donc ne demandant pas de contrôler les chaos d'ordre supérieur ; ce type d'inégalité a été baptisé *Inégalité de Poincaré d'ordre 2*. Cela leur a permis d'améliorer des résultats de normalité asymptotique pour des fonctionnelles en rapport avec le graphe des plus proches voisins, ou la mosaïque Poisson-Voronoï. J'ai utilisé cette inégalité dans [Lac19] pour établir des résultats de normalité asymptotique en rapport avec les excursions de champs shot-noise, ou d'autres fonctionnelles plus générales qui n'admettent pas de rayon de stabilisation. A cette occasion, j'ai développé un outil spécifique pour montrer des bornes inférieures de variance pour fonctionnelles poissoniennes stationnaires ; un problème en général disjoint, mais nécessaire à la preuve de la normalité asymptotique. En parallèle, et en nous appuyant sur des travaux de Chatterjee [Cha08], avec Giovanni Peccati nous avons établi le même type d'inégalité que [LPS16] mais dans le cadre binomial [LP17], c'est-à-dire quand le processus poissonien est remplacé par un ensemble de variables i.i.d. de même intensité (voir figure 3). Nous avons appliqué ce résultat à l'approximation Voronoï d'un ensemble, ou la couverture d'un domaine par un modèle booléen. Avec Matthias Schulte et Joe Yukich [LSY19], nous avons ensuite rapproché ces deux résultats (poissonien et binomial) dans un cadre général de géométrie aléatoire, pour améliorer ou établir les vitesses de convergence de fonctionnelles stabilisantes. Cela nous a permis notamment de donner la vitesse de convergence (ou de l'améliorer) dans les problèmes d'enveloppe convexe aléatoire de grands échantillons dans un ensemble convexe, comme décrit quelques pages plus haut.

Plan

Dans l'introduction de la première partie, on introduit diverses notations relatives aux mesures ponctuelles marquées qui vont constituer l'aléa des variables que l'on étudie. Dans le Chapitre 1, on expose la méthode de Stein et les opérateurs de Malliavin dans le cadre poissonien, et ses applications aux U-statistiques. On donne dans le Chapitre 2 des méthodes pour borner inférieurement la variance. Dans le chapitre 3, on donne les *inégalités de Poincaré d'ordre* 2 dans les cadres poissonien et binomial, et leur application aux fonctionnelles stabilisantes. Le chapitre 4 est dédié aux applications que mes co-auteurs et moi-même avons déduites des résultats pré-cités, sur les mosaïques de Voronoï, les enveloppes convexes de points aléatoires, les excursions de champ shot-noise, et d'autres problèmes. Enfin, le chapitre 5 contient des théorèmes limites obtenus par d'autres méthodes.

Seconde partie : Mesures marginales pour champs aléatoires et processus ponctuels

Cette partie traite des propriétés mathématiques des marginales des modèles de géométrie aléatoires, qui peuvent être vues comme les projections fini-dimensionnelles de la loi de ces modèles. Par exemple, pour un ensemble aléatoire F, la fonction

$$\ell_{\mathsf{F}}^k : (x_1, \ldots, x_k) \mapsto \mathbf{P}(x_1, \ldots, x_k \in \mathsf{F}),$$

est une marginale d'ordre k. Pour un processus ponctuel η, la marginale d'ordre k est une mesure définie informellement pour des x_i distincts par

$$\mu_\eta^k : (x_1, \ldots, x_k) \mapsto \mathbf{P}(dx_1 \in \eta, \ldots, dx_k \in \eta).$$

La marginales d'ordre 2 est souvent désignée par le terme *covariance*, ou *corrélation*, en fonction de la renormalisation et du recentrage appliqués, quel que soit le type de modèle. Elle fournit une description mathématique très incomplète, mais elle est tout de même souvent utilisée dans les applications comme outil exhaustif de description des propriétés statistiques du modèle. Elle renseigne en particulier sur la régularité et les propriétés de dépendance à grande distance.

Mon travail sur les marginales de mesures aléatoires a commencé à la fin de ma thèse en 2010, lors d'un séjour à l'Université de Bern en compagnie d'Ilya Molchanov. Celui-ci m'a soumis le problème dit S^2 de *réalisabilité* pour ensembles aléatoires : étant donné une fonction $\ell^2(x, y)$, existe-t-il un ensemble aléatoire F dont ℓ^2 est la marginale d'ordre 2, i.e. $\ell^2 = \ell_{\mathsf{F}}^2$? Au même moment, Kuna, Lebowitz et Speer [KLS11] ont développé une méthode abstraite pour traiter de problèmes de réalisabilité dans le cadre des processus ponctuels, avec des vues en mécanique statistique. Nous avons pu reprendre leur argument pour donner dans [LM15] une autre réponse au problème général de réalisabilité. Reprenons le problème S^2 pour expliquer notre approche.

Pour répondre à ce problème, il faut donc construire une mesure de probabilité μ, sur la classe des ensembles mesurables, dont ℓ^2 est la marginale d'ordre 2. Les propriétés d'additivité que μ doit satisfaire imposent d'importantes restriction algébriques sur ℓ^2, comme la nécessité d'être de type positif. La σ-additivité de μ concerne en général des propriétés liées à la régularité, ou la continuité de ℓ^2, mais est difficile à désimbriquer du problème global de réalisabilité. C'est ce travail que nous avons réalisé avec Ilya Molchanov : pour découpler le problème de réalisabilité, il peut être nécessaire de l'accompagner d'une condition supplémentaire de *régularité*, comme la non-concentration des points du processus ponctuel, ou la finitude du périmètre de l'ensemble aléatoire. Cela nous a permis de donner une réponse satisfaisante au problème de réalisabilité pour processus ponctuels. Ce n'est qu'après mon recrutement à l'Université Paris Descartes que j'ai pu apporter avec Bruno Galerne [GL15] une réponse similaire dans le cadres des ensembles aléatoires. Il s'est avéré que le bon cadre de travail était celui des *e*nsembles mesurables aléatoires, que l'on a donc développé, et que la bonne condition de régularité était celle du périmètre variationnel fini. J'ai également travaillé sur la question purement algébrique du problème, et sur un algorithme permettant de disqualifier rapidement des marginales candidates à la réalisabilité [Lac15]. Le problème dual, qui est de valider la réalisabilité d'une marginale candidate, semble être insoluble en temps fini si l'on ne dispose complètement d'une mesure aléatoire qui la réalise [DL97].

Dans un second temps, à partir de 2015, j'ai travaillé sur les liens entre la marginale d'ordre 3 d'un ensemble aléatoire de $\mathbb{R}^2$, et sa topologie [Lac18b]. J'ai en particulier montré que pour un ensemble F borné de frontière $\mathscr{C}^{1,1}$, pour $\varepsilon > 0$ suffisamment petit, sa caractéristique d'Euler $\chi(\mathsf{F})$ s'écrit

$$\begin{aligned}\chi(\mathsf{F}) = \varepsilon^{-2} \int_{\mathbb{R}^2} &[\mathbf{P}(x \in \mathsf{F}, x + \varepsilon\mathbf{u}_1 \notin \mathsf{F}, x + \varepsilon\mathbf{u}_2 \notin \mathsf{F}) \\ &- \mathbf{P}(x \notin \mathsf{F}, x - \varepsilon\mathbf{u}_1 \in \mathsf{F}, x - \varepsilon\mathbf{u}_2 \in \mathsf{F})]dx\end{aligned}$$

(la fonction $(x, y, z) \mapsto \mathbf{P}(x \in \mathsf{F}, y \notin \mathsf{F}, z \notin F)$ est une marginale d'ordre 3 au même titre que ℓ^3_{F}, et s'écrit explicitement comme combinaison linéaire des $\ell^k_{\mathsf{F}}, 0 \leqslant k \leqslant 3$). Ce travail fait largement écho, dans le cadre continu, à des pratiques de morphologie mathématique et d'analyse d'image dans un cadre d'approximation discrète (voir [Ser82]). J'ai appliqué ces résultats à l'excursion d'un champ aléatoire f dans [Lac18a], c'est-à-dire l'ensemble aléatoire

$$\mathsf{F}_u = \{x \in \mathbb{R}^2 : \mathsf{f}(x) \geqslant u\}, u \in \mathbb{R}.$$

Pour échanger espérance et intégrale sur $\mathbb{R}^2$, il a fallu contrôler le nombre de composantes connexes $\mathsf{N}(\mathsf{F}_u)$ via la formule

$$\mathsf{N}(\mathsf{F}_u) \leqslant \max(\mathrm{Lip}(\mathsf{f}), \mathrm{Lip}(\partial_i \mathsf{f}), 1 \leqslant i \leqslant d)^d \int_{\mathbb{R}^d} \frac{\mathcal{L}^d(dx)}{\max(|\mathsf{f}(x)|, |\partial_i \mathsf{f}(x)|)^d},$$

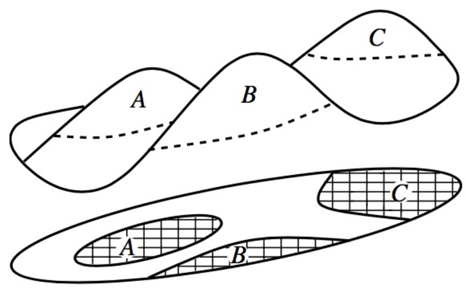
où $\mathrm{Lip}(\cdot)$ désigne la constante de Lipschitz d'une application entre deux espaces métriques. Ces résultat ont permis de relâcher légèrement les hypothèses habituelles de densités et de régularité pour l'expression de la caractéristique d'Euler moyenne d'excursions aléatoires.

Plan

On présente au chapitre 6 les travaux sur la topologie des ensembles aléatoires lisses, et au chapitre 7 ses applications pour excursions de champs aléatoires, et en particulier gaussiens.

Le chapitre 8 introduit le problème de réalisabilité, et la réponse que nous y avons apporté avec Ilya Molchanov, d'abord dans un cadre général, puis pour les processus

Figure 5 – Fonction multidimensionnelle, et une de ses excursions. A,B et C représentent les composantes connexes de l'excursion. [AT07]



ponctuels répulsifs. Enfin on explique au chapitre 9 le traitement du problème S^2 de réalisabilité pour ensembles aléatoires en utilisant ce résultat.

Le chapitre 13, de nature plus statistique, concerne un travail différent [**MCRLRMB**], celui d'estimer la marginale d'ordre 1 d'un processus ponctuel à l'aide de sa mosaïque de Voronoï.

Travaux

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Notation et cadre général

Dans tout le mémoire, $(\Omega, \mathscr{O}, \mathbf{P})$ est une espace probabilisé, où vivent tous les objets aléatoires considérés.

On introduit un espace sous-jacent $\mathbb{X}$, qui est souvent un sous ensemble d'un espace euclidien. Si $\mathbb{X}$ est muni d'une topologie, il sera équippé de la tribu borélienne $\mathscr{X} = \mathscr{B}(\mathbb{X})$, et éventuellement d'une mesure μ. Si $\mathbb{X} \subseteq \mathbb{R}^d$, il sera implicitement équippé de la mesure de Lebesgue d-dimensionnelle $\mu = \mathcal{L}^d$.

Première partie

Théorèmes limites en géométrie aléatoire

Introduction and notation

The binomial process consists in n independent points uniformly distributed in some measurable domain of $\mathbb{R}^d$ with finite Lebesgue measure, where $n \in \mathbb{N}$. It is perhaps the most natural and basic model of a random set of points. For a large random set of points, it might be more natural to consider that the number of points itself is also unknown, hence random. In this case, it is more appropriate to use a Poisson process. Furthermore, when the number of points goes to infinity, after proper rescaling, the binomial process locally converges to the Poisson process. The Poisson process, allowed to have an infinite number of points, is a cornerstone of theoretical and applied probability.

Counting measures

Let us introduce some notation related to point processes. The reader is referred to [DV88a] for a detailed exposition of this topic.

Formally, a point process in the underlying measurable space $(\mathbb{X}, \mathscr{X})$ is a counting measure, i.e. a measure on $\mathscr{X}$ taking only integer values. When a counting measure ζ is simple, i.e. $\zeta(\{x\}) \in \{0,1\}$ for all $x \in \mathbb{X}$, ζ is unambiguously associated with its support set $\{x : \zeta(\{x\}) = 1\}$. We will formally treat ζ as a set rather than a measure, even if some concepts applied to it, such as integration, refer to the associated measure. Call $\mathbf{N} = \mathbf{N}(\mathbb{X})$ the space of counting measures (viewed as point configurations), which basic elements are the Dirac masses $\delta_x, x \in \mathbb{X}$. Endow $\mathbf{N}$ with the σ-algebra $\mathscr{B}(\mathbf{N})$ generated by the counting functionals

$$\varphi_A : \zeta \in \mathbf{N} \mapsto |\zeta \cap A| \in \mathbb{N} \cup \{\infty\}, A \in \mathscr{X}.$$

For $\lambda \geqslant 0$, denote by $\mathcal{P}(\lambda)$ the Poisson law with parameter λ on $\mathbb{N} \cup \{\infty\}$, allowing for the degenerate cases $\lambda = 0$ or $\lambda = \infty$. Let μ be a σ-finite measure on $(\mathbb{X}, \mathscr{X})$. The Poisson process with intensity measure μ is the unique point process η of $(\mathbf{N}, \mathscr{B}(\mathbf{N}))$ such that

- for every $A \in \mathscr{X}$, $\eta(A) \stackrel{(d)}{=} \mathcal{P}(\mu(A))$
- For disjoint $A_1, \dots, A_m$ of $\mathscr{X}$, $\eta(A_1), \dots, \eta(A_m)$ are independent.

For existence and further properties, see [PR16]. The law of η is denoted by $\mathcal{P}(\mu)$.

Marked processes

In many applications, the carrier space can be decomposed as $\mathbb{X} = \mathbb{X}_g \times \mathbb{M}$ where $(\mathbb{X}_g, \mathscr{X}_g, \mu_g)$, the *ground space*, represents the spatial or geometric part, and $(\mathbb{M}, \mathscr{M}, \nu)$ is the *marks space*, representing additional parameters of the points. $\mathbb{X}$ is endowed with the product σ-algebra $\mathscr{B}(\mathbb{X}) = \mathscr{X}_g \otimes \mathscr{M}$, the product measure $\mu = \mu_g \otimes \nu$, and the corresponding space of configurations $(\mathbf{N}(\mathbb{X}), \mathscr{B}(\mathbf{N}(\mathbb{X})))$. The reader not familiar with such functionals can simply consider the case where $\mathbb{M}$ is a singleton, and ignore all mark related notation and considerations.

In the marked setup, spatial operations can still be performed on $\xi \in \mathscr{B}(\mathbb{X})$, these operations being implicitly applied to the spatial components of elements of ξ:

$$\begin{aligned} a\xi =& \{(ax, m) \ : \ (x, m) \in \xi\}, a \geqslant 0, \text{ if } \mathbb{X} \subset \mathbb{R}^d, \\ \xi + y =& \{(x + y, m) \ : \ (x, m) \in \xi\}, y \in \mathbb{R}^d, \text{ if } \mathbb{X} \subset \mathbb{R}^d, \\ \xi \cap A =& \xi \cap (A \times \mathbb{M}) \text{ for } A \subset \mathbb{X}_g. \end{aligned}$$

If $\mathbb{X}_g$ is endowed with a semi-metric d, also denote, for $x = (y, m), x' = (y', m') \in \mathbb{X}$, $\bar{\mathsf{d}}(x, x') = \mathsf{d}(y, y')$, and $B_{\bar{\mathsf{d}}}(x, r) = B_{\mathsf{d}}(y, r) \times \mathbb{M}$, where $B_{\mathsf{d}}(y, r)$ is the ball with center y and radius r in $(\mathbb{X}_g, \mathsf{d})$.

Chapter 1

The Stein-Malliavin method in the Poisson framework: a brief introduction

Stein's method is a general principle aimed at proving and quantifying a limit theorem towards a random variable with a given law ν. The basic ingredient on the real line is to find a class of functions $\mathscr{F} : \mathbb{R} \to \mathbb{R}$ and a differential operator L such that a variable U has law ν iff for every $f \in \mathscr{F}$

$$\mathbf{E}(\mathsf{L}f(U)) = 0. \tag{1.1}$$

For instance, let $\gamma(dx) = (2\pi)^{-1/2}\exp(-x^2/2)dx$ be the the standard Gaussian distribution on $\mathbb{R}$. Then $\mathscr{F} = \mathscr{C}_c^1(\mathbb{R})$ and $\mathsf{L}f(x) = f(x) - xf'(x)$ characterize γ in the sense of (1.1). The general principle of the method is to quantify the assertion that, for some real variable V, the more the quantity $\mathbf{E}[\mathsf{L}f(V)]$ is close to 0 for many $f \in \mathscr{F}$, the more the law of V is close to ν.

To that end, choose another class of functions $\mathscr{H}$ where for $h \in \mathscr{H}$, one can control the solution f_h of the differential equation $\mathsf{L}f_h = h - \int_{\mathbb{R}} h(u)\nu(du)$ and its derivatives at a sufficiently high order. It is immediate that, if $\mathscr{H}$ is sufficiently rich so that $d_{\mathscr{H}}(U,V) := \sup_{h\in\mathscr{H}} |\mathbf{E}h(U) - \mathbf{E}h(V)|$ is a distance between the laws of random variables U and V, then for V with law ν,

$$d_{\mathscr{H}}(U,V) = \sup_{h\in\mathscr{H}} |\mathbf{E}\mathsf{L}f_h(U)|. \tag{1.2}$$

The rest of the method consists in bounding the right hand term by appropriate tools, quite often involving the chain rule and integration by parts on an appropriate space. These tools shall greatly depend on the type of random input in U (a Gaussian measure, a Poisson process, n i.i.d variables), and rely on differential operators introduced on the space where lives the input process. The novelty of the approach developed in the Gaussian realm by Nourdin, Peccati, Nualart, and their co-authors has been to introduce this differential structure via Malliavin calculus.

The distances we will investigate here consist in the Wasserstein distance, for which $\mathscr{H}_W$ is the set of 1-Lipschitz functions $\mathbb{R} \to \mathbb{R}$, and the Kolmogorov distance, for which $\mathscr{H}_K = \{I_t; t \in \mathbb{R}\}$ where $I_t(x) = \mathbf{1}_{\{x\leqslant t\}}, x \in \mathbb{R}$. In other words, given two random variables U, V, (we note for simplicity $\mathsf{d}_W = \mathsf{d}_{\mathscr{H}_W}, \mathsf{d}_K = \mathsf{d}_{\mathscr{H}_K}$)

$$\mathsf{d}_W(U,V) = \sup_{h \text{ 1-Lipschitz}} (\mathbf{E}[h(U)] - \mathbf{E}[h(V)])$$
$$\mathsf{d}_K(U,V) = \sup_{t\in\mathbb{R}} |\mathbf{P}(U \leqslant t) - \mathbf{P}(V \leqslant t)|\,.$$

To illustrate the method, let us a give Stein's lemma in the context of normal approximation with Kolmogorov distance:

Lemma 1 ([LP17])**.** *Let $N \sim \mathscr{N}(0,1)$. Then for any random variable U,*

$$\mathsf{d}_K(U,N) \leqslant \sup_f |\mathbf{E}\mathsf{L}f(U)|,$$

where the supremum runs over functions f which are continuous on $\mathbb{R}$, differentiable on $\mathbb{R}\setminus\{t\}$ for some $t\in\mathbb{R}$, with $\|f'(x)\| \leqslant 1, x \in \mathbb{R}\setminus\{t\}$, and satisfy some sort of second order inequality: for all $x, y \in \mathbb{R}$,

$$|f(x+y) - f(x) - yf'(x)| \leqslant \frac{y^2}{2}\left(|x| + \frac{\sqrt{2\pi}}{4}\right) + \underbrace{|y|\left(\mathbf{1}_{\{t\in[x,x+y)\}} + \mathbf{1}_{\{t\in[x+y,x)\}}\right)}_{=y(\mathbf{1}_{\{t\in[x,x+y)\}} - \mathbf{1}_{\{t\in[x+y,x)\}})},$$

with the convention $f'(t) = tf(t) + 1 - \mathbf{P}(N \leqslant t)$.

This result is the consequence of (1.2) and the fact that for $t \in \mathbb{R}$, the differential equation

$$\mathsf{L}f(x) := f'(x) - xf(x) = I_t(x) - \mathbf{P}(N \leqslant t), x \in \mathbb{R},$$

admits a solution f_t which is like in Lemma 1.

This lemma is used in Section 3 to derive normal approximation for binomial functionals. Stein's method efficiency essentially resides in the fact that the derivatives of the ODE solution f_t are bounded.

1.1 Functionals

For $\mathbf{N}' \subset \mathbf{N}(\mathbb{X})$, let $\mathscr{F}(\mathbf{N}')$ be the space of functionals $\mathbf{N}' \to \mathbb{R}\cup\{\pm\infty\}$. Call $\mathbf{N}^f \subset \mathbf{N}$ the space of finite subsets of $\mathbb{X}$. When the random input is a binomial process, the number of points is a.s. finite and any functional defined on $\mathbf{N}^f$ can be applied to a binomial process.

A Poisson measure η might have infinitely many points with probability 1, hence it is not clear what functionals are properly defined on η a.s. A typical example is the functional

$$F(\zeta) = \sum_{x\in\zeta} \frac{\sin(\|x\|)}{1 + \|x\|^{d+1}}, \zeta \subset \mathbb{R}^d.$$

F is not properly defined on all $\mathbf{N}(\mathbb{R}^d)$, but $F(\eta)$ is well defined a.s. for $\eta \sim \mathcal{P}(\mathcal{L}^d)$. Similar considerations affect the definition of shot noise fields at Section 4.1.

Definition 2. A couple $(F, \mathbf{N}_F)$ is said to be η-admissible if $\mathbf{N}_F \in \mathscr{B}(\mathbf{N}(\mathbb{X}))$ is such that $\mathbf{P}(\eta \in \mathbf{N}_F) = 1$, $\mathbf{N}_F$ is stable under the union with a finite set, and F is a mapping from $\mathbf{N}$ to $\mathbb{R}\cup\{\pm\infty\}$.

In many works, $\mathbf{N}_F$ is implicitly defined as the subspace of $\mathbf{N}(\mathbb{X})$ over which F is well defined.

1.2 Stochastic integrals and Malliavin operators

The results presented in this section are proved, developed and further commented upon in [Pec+10; PR16]. Let $\eta \sim \mathcal{P}(\mu)$ on $(\mathbb{X}, \mathscr{X})$, where μ is a Radon measure without atoms. The natural stochastic integral on a random Poisson measure η is defined a.s. for $f \in \mathscr{L}^1(\mu)$ by

$$\eta(f) = \sum_{x \in \eta} f(x).$$

Denote by $\mathscr{L}^2(\mu^q) := \mathscr{L}^2(\mathbb{X}^q, \mu^q)$ the space of square-integrable functions with respect to μ^q. Higher order integrals are defined by, for $f \in \mathscr{L}^1(\mu^q), q \geqslant 1$,

$$\eta^q(f) = \sum_{(x_1,\ldots,x_q) \in \eta^q} f(x_1, \ldots, x_q). \tag{1.3}$$

Such integrals are also called U-statistics and arise naturally in many problems. If they can be considered as the polynomials of $\mathscr{L}^2(\eta)$, the class of square integrable variables measurable with respect to η, then we explicit below a natural orthogonal basis of this class of polynomials. It will constitute the core of the Malliavin calculus performed on $\mathscr{L}^2$ Poisson functionals.

The multiple Wiener-Itô integrals are the multiple integrals with respect to $\hat{\eta} := \eta - \mu$ the compensated version of η, after removing diagonal points: for $q \geqslant 1, f \in \mathscr{L}^2(\mu^q)$

$$I_q(f) = \int_{\mathbb{X}^{(q)}} f(x_1, \ldots, x_q) d\hat{\eta}^q(x_1, \ldots, x_q),$$

where $\mathbb{X}^{(q)}$ is the set of q-tuples of distinct points of $\mathbb{X}$. The centering of the measure and the Campbell-Mecke formula confer the following orthogonal structure: for $q, m \geqslant 1, f \in \mathscr{L}^2(\mu^q), g \in \mathscr{L}^2(\mu^m)$,

$$\mathbf{E}(I_q(f) I_m(g)) = q! \delta_{m=q} \langle f, g \rangle_{\mu^q},$$

where $\langle \cdot, \cdot \rangle_{\mu^q}$ is the standard scalar product on $\mathscr{L}^2(\mu^q)$. Then every $F \in \mathscr{L}^2(\eta)$ admits the decomposition in the $\mathscr{L}^2$ sense

$$F = \sum_{q \geqslant 0} I_q(f_q) \tag{1.4}$$

for some kernels $f_q \in L^2_s(\mu^q)$. We explicit below each f_q up to a μ^q-negligible set.

Let us now explain how the structure induced by this decomposition adapts naturally to Malliavin calculus on the Poisson space. Given $(F, \mathbf{N}_F)$ η-admissible and $\zeta \in \mathbf{N}_F$, the first-order Malliavin derivative of F is defined by

$$\mathsf{D}_x F(\zeta) = F(\zeta \cup \{x\}) - F(\zeta), x \in \mathbb{X}.$$

This operator transforms random variables into random functions. Higher order derivatives are defined by iterating this operator: for $x_1, \ldots, x_q \in \mathbb{X}$, $q \geqslant 1$

$$\mathsf{D}^q_{x_1,\ldots,x_q} F(\zeta) = \mathsf{D}_{x_q}(\mathsf{D}^{q-1}_{x_1,\ldots,x_{q-1}} F)(\zeta). \tag{1.5}$$

This operator is symmetric : the result remains unchanged if the order of the x_i is modified. The second-order derivative $\mathsf{D}^2_{x,y} F$ will play a special role in Chapter 3 as a measure of the influence between x and y for the functional F.

Not every $\mathscr{L}^2$ functional admits a derivative. In the sequel, denote by $\mathscr{D}(\eta)$ the class of functionals $F \in \mathscr{L}^2(\eta)$ such that $\int_{\mathbb{X}} \mathbf{E}(\mathsf{D}_x F^2)\mu(dx) < \infty$. In terms of the decomposition (1.4), this condition is equivalently expressed as $\sum_{q \geqslant 1} q \cdot q! \|f_q\|^2_{\mathscr{L}^2(\mu^q)} < \infty$. Reasoning chaos by chaos, it is easy to prove that for $F \in \mathscr{D}(\eta)$

$$\mathsf{D}_x F = \sum_{q \geqslant 1} q I_{q-1}(f_q(x, \cdot)), x \in \mathbb{X}, \tag{1.6}$$

where $f_q(x, \cdot)$ is the element of $\mathscr{L}^2(\mu^{q-1})$ which to $x_1, \ldots, x_{q-1}$ associates $f(x, x_1, \ldots, x_{q-1})$. Iterating this principle, one can prove that for $F \in \mathscr{L}^2(\eta)$, for μ^q-a.e. $x_1, \ldots, x_q \in \mathbb{X}$

$$f_q(x_1, \ldots, x_q) = \frac{1}{q!}\mathbf{E}(\mathsf{D}^q_{x_1, \ldots, x_q} F(\eta)). \tag{1.7}$$

In particular, we have the variance decomposition and lower bounds

$$\mathrm{Var}(F) = \sum_{q \geqslant 0} \mathrm{Var}(I_q(f_q)) = \sum_{q \geqslant 0} q! \|f_q\|^2_{\mathscr{L}^2(\mu^q)} \geqslant \|f_1\|^2 = \int (\mathbf{E}[\mathsf{D}_x F])^2 \mu(dx). \tag{1.8}$$

See (2.2) for an analogue bound with binomial processes. This inequality provides in some cases a sharp lower bound for the variance magnitude, see [Sch16]. It can be compared to the upper bound provided by the *first order Poincaré inequality*

$$\mathrm{Var}(F) \leqslant \int_{\mathbb{X}} \mathbf{E}(\mathsf{D}_x F^2)\mu(dx).$$

In the following, we seek to establish an operator I that satisfies the integration by parts formula:

$$\mathbf{E}(FG) = \mathbf{E}\left[\langle \mathsf{D}F, \mathsf{I}G \rangle_{\mathscr{L}^2(\mu^2)}\right], \quad F \in \mathscr{D}(\eta), G \in \mathscr{L}^2(\eta). \tag{1.9}$$

The Ornstein-Uhlenbeck operator L and its inverse, named in analogy with their homonymous in the Gaussian realm, allow for such a formula. L is an operator defined for functionals F such that $\sum_{q \geqslant 1} q^2 q! \|f_q\|^2_{\mathscr{L}^2(\mu^q)} < \infty$ via

$$\mathsf{L}F = -\sum_{q \geqslant 1} q I_q(f_q).$$

Note that this is a centred variable. Its inverse $\mathsf{L}^{-1}F$ is defined on the space of centred square-integrable variables by

$$\mathsf{L}^{-1}F = -\sum_{q \geqslant 1} \frac{1}{q} I_q(f_q).$$

Using (1.6), these chaotic representations allow to prove that (1.9) is in order with

$$\mathsf{I}F := -\mathsf{D}\mathsf{L}^{-1}F.$$

Things get interesting when the operator L^{-1} can be interpreted with the help of a dynamics on the random input η, using the concept of thinning: for $s \in [0, 1]$, attach to the points of η independent Bernoulli variables $B_x, x \in \eta$ with parameter s, and let $\eta_s = \{x \in \eta : B_x = 0\}$. In other words, η_s is obtained from η by independently removing the points of η with probability s; η_s is called a $(1-s)$-*thinning* of η. It allows in particular to couple the $\eta_s, s \geqslant 0$ on the same probability space. Let then $(\eta'_s)_{s\geqslant 0}$ be an independent copy of $(\eta_s)_{s\geqslant 0}$. Define the semi-group

$$\mathsf{P}_s F(\eta) = \mathbf{E}[\eta_s \cup \eta'_{1-s} \mid \eta],\ s \in [0,1].$$

Reasoning chaos by chaos and doing an infinite $\mathscr{L}^2$ summation, it is possible to show that the operator L^{-1} and its derivative admit the dynamic representation

$$\mathsf{L}^{-1}F = -\int_0^1 s^{-1}\mathsf{P}_s\mathsf{D}F ds \tag{1.10}$$

$$-\mathsf{I}F = \mathsf{D}\mathsf{L}^{-1}F = -\int_0^1 \mathsf{P}_s\mathsf{D}F ds. \tag{1.11}$$

This result is known as Mehler's formula, it has been established by Privault [Pri09] and by Last, Peccati and Schulte [LPS16] to establish second order Poincaré inequalities, see Chapter 3.

1.3 Bounds on the distance with the Gaussian law

By combining Stein's lemma with the integration by parts formula 1.9, Peccati, Solé, Utzet and Taqqu have obtained the following abstract bound for Poisson functionals:

Theorem 3 (Th. 3.1 in [Pec+10])**.** *Let F be a centred variable from $\mathscr{D}(\eta)$, $N \sim \mathscr{N}(0,1)$. Then*

$$\mathsf{d}_W(F,N) \leqslant \sqrt{\mathbf{E}\left[\left(1-\langle \mathsf{D}F, -\mathsf{D}\mathsf{L}^{-1}F\rangle_{\mathscr{L}^2(\mu)}\right)^2\right]} + \int_{\mathbb{X}} \mathbf{E}\left[|\mathsf{D}_zF|^2|\mathsf{D}_z\mathsf{L}^{-1}F|\right]\mu(dz). \tag{1.12}$$

The Stein lemma used to obtain this expression is simpler than Lemma 1, as the latter is tailored to deal with Kolmogorov distance, which conveys more terms than Stein's lemma for Wasserstein distance.

The explicit chaotic expressions of the Malliavin operators allow to apply this formula to the building blocks of $\mathscr{L}^2(\eta)$, i.e. the multiple integrals, also called Poisson chaoses, and the U-statistics, see below. For functionals who don't admit a finite expansion, that might turn out to be more difficult. In [HLS16], Hug, Last and Schulte use Steiner's formula and this decomposition to prove central limit theorems for intrinsic volumes of Boolean models. These results are hard to reproduce in a general context, as it is difficult to efficiently estimate those kernels without other tools or assumptions at our disposal. See Chapter 3 to overcome this difficulty in a different way.

1.3.1 Contractions and 4th moment theorem

One way to exploit this formula is to write all the Malliavin operators in terms of how they act on the kernels in the Wiener-Itô decomposition (1.4), and plug back these expressions in (1.12). Then one has to compute the chaotic decomposition of products of multiple integrals and estimate the scalar products involved. This can be done with the help of the product formula: for $f \in \mathscr{L}^2_s(\mu^p), g \in \mathscr{L}^2_s(\mu^q)$,

$$I_p(f)I_q(g) = \sum_{r=0}^{\min(p,q)} r!C_p^rC_q^r\sum_{l=0}^{r} C_r^l I_{q+p-r-l}(\widetilde{f\star_r^l g})$$

where $(f,g) \mapsto f \star_r^l g$ is a bi-linear operator called *contraction* of f and g of indexes r and l. It is formally defined on $\mathbb{X}^{p+q-r-l}$ by

$$f \star_r^l g(x_1, \ldots, x_{r-l}, y_1, \ldots, y_{p-r}, z_1, \ldots, z_{q-r})$$
$$= \int_{\mathbb{X}^l} f(t_1, \ldots, t_l, x_1, \ldots, x_{r-l}, y_1, \ldots, y_{p-r}) g(t_1, \ldots, t_l, x_1, \ldots, x_{r-l}, z_1, \ldots, z_{q-r}) dt_1 \ldots dt_l,$$

but this expression does not always make sense, see the technical requirements in [Pec+10] or [LP13a]. The operation $\widetilde{f \star_r^l}$ is a symmetrization of $f \star_r^l$ with respect to its $p+q-r-l$ arguments.

A first way to exploit contractions is the following result, focusing on variables with a finite decomposition and non-negative kernels, which states that asymptotic normality is essentially equivalent to the convergence of the first four moments to those of a Gaussian variable.

Theorem 4 ([LP13a], Th. 3.12)**.** *Fix $k \geqslant 1$. Let $F_n = \sum_{i=1}^k I_{q_i}(f_{i,n})$ where the $q_i \geqslant 1$ are strictly increasing and $f_{i,n} \in L_s^2(\mu^{q_i})$ and $f_{i,n} \geqslant 0$. Assume also that $(F_n^4)_{n \geqslant 1}$ is uniformly integrable. Then $F_n \to N$ in law as $n \to \infty$ iff $\mathbf{E}(F_n^4) - 3\mathrm{Var}(F_n)^2 \to 0$.*

This type of results is called a *4th moment theorem*, it considerably simplifies the method of moments, where it must be checked that all the moments converge to that of a Gaussian. It has been since then much improved by Dobbler and Peccati [DP18], in particular they relaxed the non-negativity assumption on the kernel.

1.3.2 U-statistics

Denote by $\eta_k^{\neq}$ the set of distinct k-tuples of points of $\mathbb{X}$. Let $h \in \mathscr{L}^1(\mu^q)$ symmetric and

$$F_h(\eta) := \sum_{\mathbf{x}=(x_1,\ldots,x_k) \in \eta_k^{\neq}} h(\mathbf{x}).$$

A variable of this form is called a *U-statistic*. It arises for instance in statistics related to the Gilbert graph on η: with $h(x,y) = \mathbf{1}_{\{\mathsf{d}(x,y) \leqslant 1\}}$, $F_h(\eta)$ counts the number of edges in the graph which vertex set is η and an edge is formed by any two points at distance less than 1. It applies in numerous other instances, such as the count of k-flats intersecting a given convex body, other statistics related to the Gilbert graph, intersection processes, or estimation of the Sylvester constant in the random convex hull problem; see [RS13],[LR16] and references therein.

The efficiency of the Stein-Malliavin method in stochastic geometry has been revealed by the work of Reitzner and Schulte [RS13], where the asymptotic behaviour of U-statistics of geometric nature has been studied for two classes of U-statistics, called local U-statistics and geometric U-statistics.

A nice feature of U-statistics is that their Wiener-Itô decomposition can be easily deduced from the binomial formula $\eta^q = (\hat{\eta} + \mu)^q = \sum_{k=0}^q C_q^k \hat{\eta}^k \mu^{q-k}$: provided $F_h(\eta)$ is square-integrable, the kernel f_q of the Wiener-Itô decomposition (1.4) is given by

$$f_q(x_1, \ldots, x_q) = \mathbf{1}_{\{q \leqslant k\}} \binom{q}{k} \int_{\mathbb{X}^{k-q}} h(x_1, \ldots, x_q, x_{q+1}, \ldots, x_k) d\mu^{k-q}(x_{q+1}, \ldots, x_k). \tag{1.13}$$

One can then explicit the values of the Malliavin operators $\mathsf{D}F_h, \mathsf{D}\mathsf{L}^{-1}F_h$ and inject them into 1.12 to obtain a distance estimate with the standard Gaussian law.

In the work [LP13b], we study a general U-statistic model on $\mathbb{X} = \mathbb{R}^d$ of the form

$$F_\lambda(\eta) = \sum_{\mathbf{x}=(x_1,\dots,x_k)\in(\eta\cap\lambda^{1/d}Q)^k_{\neq}} h(\alpha_\lambda \mathbf{x}), \lambda > 0,$$

where η is a homogeneous Poisson point process with intensity 1, Q is a subset of $\mathbb{R}^d$, $\alpha_\lambda > 0$ is a rescaling factor, h is an integrable function on $(\mathbb{R}^d)^k$. One can also consider a marked kernel $h \in \mathscr{L}^1((\mathbb{R}^d \otimes \mathbb{M})^k)$ for some marks space $\mathbb{M}$. The asymptotic behaviour of α_λ as $\lambda \to \infty$ determines the regime of the asymptotic behaviour of $\tilde{F}_\lambda(\eta) := (F_\lambda(\eta) - \mathbf{E}(F_\lambda(\eta)))\mathrm{Var}(F_\lambda(\eta))^{-1/2}$. Say that h has *rapidly decaying projections* if it is spatially stationary and the projections

$$h_q(x_1,\dots,x_q) = \int_{(\mathbb{R}^d\times\mathbb{M})^{k-q}} |h(x_1,\dots,x_k)| d\mu^{k-q}(x_{q+1},\dots,x_k), \quad x_1,\dots,x_q \in \mathbb{R}^d \times \mathbb{M},$$

decay sufficiently fast away from the diagonal, see [LP13b] for a precise statement.

The quantity $v_\lambda = \alpha_\lambda^{-d}$ gives the average number of points interacting with a typical point of η through the kernel h.

Theorem 5. *Let the previous notation prevail. Let h be a stationary symmetric kernel with rapidly decaying projections. Then if the kernels f_0, f_q are not identically equal to 0, for some $C_1, C_2, C_3 > 0$, for λ sufficiently large,*

$$C_1 \leqslant \frac{\mathrm{Var}(F_\lambda(\eta))}{\lambda v_\lambda^{2k-2} \max(1, v_\lambda^{-k+1})} \leqslant C_2$$

and

$$\max(\mathsf{d}_K(\tilde{F}_\lambda, N), \mathsf{d}_W(\tilde{F}_\lambda, N)) \leqslant C_3 \lambda^{-1/2} \max(1, v_\lambda - k + 1)^{1/2}.$$

Depending on the asymptotic behavior of v_λ, we can identify four different regimes. The n-th chaos refers to the n-th term in the decomposition (1.4). Remark that for a U-statistics of order k, in virtue of (1.13), the kernels of order $> k$ vanish.

1. **Long interactions**: $v_\lambda \to \infty$, CLT at speed $\lambda^{-1/2}$, the first chaos dominates the variance (*geometric U-statistics*).
2. **Constant size interactions**: $v_\lambda \to c > 0$, CLT at speed $\lambda^{-1/2}$, all chaoses have the same order of magnitude (*local U-statistics*).
3. **Small interactions**: $v_\lambda \to 0, v_\lambda^{-k+1} \to \infty$, CLT at speed $(\lambda v_\lambda^{-k+1})^{-1/2}$, higher order chaoses dominate. In the case of random graphs ($k = 2$), the corresponding bound in $(\lambda v_\lambda)^{-1/2}$ has been obtained in [LP13b].
4. **Rare interactions**: $\lambda v_\lambda^{-k+1} \to c < \infty$, the bound does not converge to 0. In the case $k = 2$, it has been shown in [LP13b] that there is no CLT but a Poisson limit in the case $c > 0$ (see Chapter 6 in [PR16] for more on Poisson limits).

Chapter 2

Variance bounds

Before establishing a limit law for a sequence of random variables, one has to know the right order of magnitude of the variance, if any. Giving upper and lower bounds are in general two different tasks. While the upper bound is often settled through integrability or summability of the reduced covariance function, the lower bound usually requires an ad-hoc method, where the specific geometric work might be more involved.

In the geometric framework, we refer the reader to [PY01, Theorem 2.1] for a general method to prove variance lower bounds for stabilising functionals. We present below two other methods, one in the Poisson framework, adapted to stationary functionals not necessarily admitting a stabilisation radius, and another one in the binomial framework, based on the Hoeffding decomposition.

2.1 Lower bound with orthogonal decompositions

A $\mathscr{L}^2$ Poisson functional has the variance lower bound (1.8), provided by the Wiener-Itô orthogonal decomposition. We provide here an analogous orthogonal decomposition and variance lower bound for binomial functionals. Let $n \geqslant 1$, $\xi_n = \{X_1, \ldots, X_n\}$ i.i.d. variables with law μ on $\mathbb{X}$, and $F(\xi_n)$ a $\mathscr{L}^2$ variable. The classical theory of Hoeffding decompositions for functions of independent random variables (see e.g. [15, 29]) implies that $F(\xi_n)$ admits a unique decomposition of the type

$$F(\xi_n) = \mathbf{E}(F(\xi_n)) + \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \varphi_{i_1, \ldots, i_k}(X_{i_1}, \ldots, X_{i_k}) \tag{2.1}$$

where $\mathbf{E}[\varphi_{i_1,\ldots,i_k}(X_{i_1}, \ldots, X_{i_k}) \mid X_{j_1}, \ldots, X_{j_m}] = 0$ for any strict subset $\{j_1, \ldots, j_m\} \subset \{i_1, \ldots, i_k\}$. Define the difference operator for $1 \leqslant i \leqslant n$,

$$\Delta_i F(x, x') = F(\{x_1, \ldots, x_n\}) - F(\{x_1, \ldots, x_i', \ldots, x_n\}),\ x = (x_1, \ldots, x_n) \in \mathbb{X}^n, x' = (x_1', \ldots, x_{,n}') \in \mathbb{X}^n.$$

It turns out [LP17, Theorem 2.2] that the $\varphi_{i_1,\ldots,i_k}$ admit a representation via an independent copy $\xi_n' = \{X_1', \ldots, X_n'\}$ of ξ_n and the Δ_i:

$$\varphi_{i_1,\ldots,i_k}(X_{i_1}, \ldots, X_{i_k}) = \mathbf{E}\left[\Delta_{i_1} \ldots \Delta_{i_k} F(\xi_n', \xi_n) \mid \xi_n\right].$$

This relation bears a strong similarity with (1.7), hence (2.1) can be seen as the analogue in the binomial context of the Wiener-Itô decomposition (1.4), and its terms are indeed orthogonal. Hence we have the variance lower bound

$$\mathrm{Var}(F) \geqslant \sum_{i=1}^{n} \mathbf{E}\left[(\mathbf{E}[\Delta_i F(\xi_n', \xi_n) \mid \xi_n])^2\right] = \int_{\mathbb{X}} (\mathbf{E}\left[F(\xi_n) - F(\{x, X_2, \dots, X_n\})\right])^2 n d\mu(x), \tag{2.2}$$

analogue to (1.8). Here again it can be compared to a Poincaré-like inequality, the *Efron-Stein* inequality:

$$\mathrm{Var}(F) \leqslant \frac{1}{2} \int_{\mathbb{X}} \mathbf{E}\left[(F(\xi_n) - F(\{x, X_2, \dots, X_n\}))^2\right] n d\mu.$$

This lower bound is sharp (in magnitude) in the asymptotic study of binomial Voronoï set approximation, see [LP17].

2.2 Variance asymptotics for stationary functionals

In this section we give upper and lower bounds for a special class of functionals enjoying some additivity and stationarity on $\mathbb{X}_g = \mathbb{R}^d$. We consider an auxiliary marks space $(\mathbb{M}, \mathscr{M}, \nu)$, with $\mathbb{X} = \mathbb{X}_g \times \mathbb{M}$ (see Section), the purely geometric framework $\mathbb{X} = \mathbb{X}_g$ is equivalent to taking $\mathbb{M}$ as a singleton. We start from a couple $(F_0, \mathbf{N}_0)$ that is η-admissible. We further assume that $\mathbf{N}_0$ is invariant under translation and multiplication by a scalar, and F_0 is stationary, i.e. invariant under translations of its spatial arguments. Define $\tilde{Q}_a = [-a/2, a/2)^d, a \geqslant 0$. We then consider the functional, for $W \subset \mathbb{Z}^d$, and $\tilde{W} = \cup_{k \in W}(k + \tilde{Q}_1)$,

$$F_{\tilde{W}}(\zeta) = \sum_{k \in W} F_0(\zeta - k), \zeta \in \mathbf{N}_0. \tag{2.3}$$

This model applies for instance to the classical framework of functionals written as sum of score contributions, as in Section 3.2, or to intrinsic volumes of Poisson shot-noise excursions, see Section 4.1. We won't state results for score functionals here, but it is explained in [Lac19] how to adapt the current results to the score functionals setup.

Observation window

Another functional of interest is

$$F'_{\tilde{W}}(\zeta) = F_{\tilde{W}}(\zeta \cap \tilde{W}), \zeta \in \mathbf{N}_0,$$

where the input is restricted to the observation window. The truncation might loosen the assumptions needed, but the induced edge effects also bring some additional complexity in the shape of hypotheses in the subsequent theorems. We won't investigate $F'_{\tilde{W}}(\zeta)$ here either, but again refer the reader to [Lac19] for specific variance and asymptotic normality results in this setup.

In many works (e.g. [PY01],[KM10, Chapter 4]), the observation windows consist in a growing family of subsets $B_n, n \geqslant 1$ of $\mathbb{R}^d$, that satisfy the Van'Hoff condition: for all $r > 0$,

$$\ell^d(\partial B_n^{\oplus r}) / \ell^d(B_n) \to 0, \tag{2.4}$$

as $n \to \infty$, where $B^{\oplus r} = \{x \in \mathbb{R}^d : d(x, B) \leqslant r\}$ for $B \subset \mathbb{R}^d$. We rather consider in this paper, like for instance in [SY01], a family $\mathscr{W}$ of bounded subsets of $\mathbb{Z}^d$ satisfying the regularity condition

$$\limsup_{W\in\mathscr{W}} \frac{|\partial_{\mathbb{Z}^d} W|}{|W|} = 0, \tag{2.5}$$

where $\partial_{\mathbb{Z}^d} W$ is the set of points of W at distance 1 from W^c, and consider a point process over $\tilde{W}$. In the large window asymptotics, condition (2.5) imposes the same type of restrictions as (2.4), and using subsets of the integer lattice eases certain estimates and is not fundamentally different.

Stabilisation hypothesis

To control the asymptotic behaviour, we must assume that a local contribution is not affected too strongly when the input is modified far away. For technical reasons, we also need that this property holds if two deterministic points or less are added to the input. In all the section, α is a real number such that $\alpha > 5d/2$.

Assumption 6. *Let $x_1, x_2 \in \mathbb{R}^d, \zeta \subset \{x_1, x_2\}, \eta' := \eta \cup \zeta$. Assume there is C_0 not depending on x_1, x_2 such that*

$$\mathbf{E}\left[|F_0(\eta') - F_0(\eta' \cap B(0,r))|^4\right] \leqslant C_0(1+r)^{-\alpha}, r \geqslant 0. \tag{2.6}$$

Non-degeneracy of the variance

It is a well known fact that for a functional of the form (2.3), if the summands have a low dependency for distant values of k's (materialized by condition (2.6)), then the variance is typically expected to behave in $|W|$ as $W \uparrow \mathbb{Z}^d$ under condition (2.4). This outcome also requires of course that the functional is not trivial or degenerate in some sense. To state the corresponding condition, introduce the notation for $0 \leqslant a \leqslant b \leqslant +\infty$, $\eta_a^b = \eta \cap \tilde{Q}_b \cap \tilde{Q}_a^c, \eta^b = \eta_0^b$.

A condition that seems necessary for the variance to be non-degenerate is that at least on a finite input and a bounded window, the functional is not trivial: for some $\delta > \rho > 0$, $\mathbf{P}(F_{\tilde{Q}_\delta}(\eta^\rho) \neq F_{\tilde{Q}_\delta}(\emptyset)) > 0$. We actually need it to hold uniformly if points are added far away from η^ρ:

Assumption 7. *There is $\gamma > \rho > 0, c > 0, p > 0$ such that for $\delta > \gamma$ arbitrarily large*

$$\mathbf{P}\left(\left|F_{\tilde{Q}_\delta}(\eta_\gamma) - F_{\tilde{Q}_\delta}(\eta^\rho \cup \eta_\gamma)\right| \geqslant c\right) \geqslant p.$$

We can now give variance estimates. The constant $\varkappa$ might vary from line to line and only depends on α and the dimension d.

Theorem 8. *Assume that Assumptions 6 (with $\zeta = \emptyset$ and $\alpha > d$) and Assumption 7 hold. Then there is $\varkappa > 0$ depending on α and d such that*

$$\mathrm{Cov}(F_0(\eta), F_0(\eta - k)) \leqslant \varkappa C_0^2(1 + \|k\|)^{-\alpha}, \quad k \in \mathbb{Z}^d, \tag{2.7}$$

$$0 < \sigma_\infty^2 := \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(F_0, F_k) < \infty.$$

For $W \subset \mathbb{Z}^d$ bounded and non-empty,

$$\left||W^{-1}|\mathrm{Var}(F_W(\eta)) - \sigma_\infty^2\right| \leqslant \varkappa C_0^2(|\partial_{\mathbb{Z}^d} W|/|W|)^{1-d/\alpha}. \tag{2.8}$$

If furthermore $\alpha > 2d$, we have an estimate on the 4-th moment

$$\mathbf{E}\left(F_W(\eta) - \mathbf{E}F_W(\eta)\right)^4 \leqslant \varkappa C_0(\mathbf{E}(F_0(\eta)^4))^{3/4}|W|^2. \tag{2.9}$$

The last bound is used in Section 3 to establish a speed of convergence to the normal law.

Recent similar results can be found in the literature, but the assumptions are of different nature, either dealing with different qualitative long range behaviour (i.e. strong stabilization in [PY01],[LSY19], see the comments after Theorem 13), or different non-degeneracy statements [LPS16], whereas Assumption 7 is a mixture of non-triviality and continuity of the functional on large inputs. Penrose and Yukich [PY01] give a condition under which the asymptotic variance is strictly positive in Theorem 2.1. The condition is that the functional is *strongly stabilising*, and that the variable

$$\Delta(\infty) := \lim_{\delta \to \infty} [F_{Q_\delta}(\eta \cup \{0\}) - F_{Q_\delta}(\eta)]$$

is non-trivial. It roughly means that for δ sufficiently large, and ρ sufficiently small,

$$\mathrm{Var}(|F_{\tilde{Q}_\delta}(\eta_\rho \cup \eta^\rho) - F_{\tilde{Q}_\delta}(\eta_\rho) \mid |\eta^\rho| = 1 \ ) > 0,$$

and this is quite close to Assumption 7 in the particular case $\rho = \gamma$. This particular case seems more delicate to deal with that when γ is much larger than ρ, because in the latter case the interaction between η^ρ and η_γ hopefully becomes small.

Similar results where the input consists of m_n iid variables uniformly distributed in $\tilde{W}_n$, with $m_n = |W_n|$, should be within reach by applying the results of [LP17], following a route similar to [LSY19].

Chapter 3

Second order Poincaré inequalities

Let F be a functional, and ζ a point configuration. One can see the first order derivative $\mathsf{D}_x F(\zeta) = F(\zeta \cup \{x\}) - F(\zeta)$ as the contribution of a point $x \in \mathbb{X}$ to the functional F within a point configuration ζ. Recall that the second order derivative is formally defined by

$$\mathsf{D}^2_{x,y} F(\zeta) = \mathsf{D}_x(\mathsf{D}_y F)(\zeta) = \mathsf{D}_y F(\zeta \cup \{x\}) - D_y F(\zeta).$$

Hence it can be seen as the measure of how the presence of x influences the contribution of y. Since the roles of x and y are symmetric, we can talk about a measure of the interaction between x and y. If $\mathsf{D}^2_{x,y} F(\zeta)$ is small in some sense for x, y far away in the appropriate metric, it might imply that the contributions to $F(\eta)$ of all the points of η are weakly dependent from one another, and hence the global behaviour is expected to be asymptotically normal for large random process ζ.

The idea that a second order difference operator controls the asymptotic normality has been materialized by Chatterjee [Cha09]. He proved that given a Gaussian vector $\xi = (X_1, \ldots, X_n)$ with covariance matrix Σ, and a $\mathscr{L}^2$ variable $F = g(X_1, \ldots, X_n)$ with g of class $\mathscr{C}^2$, we have the total variation distance

$$\mathsf{d}_{TV}(F, N) \leqslant \frac{C \|\Sigma\|^{3/2} (\mathbf{E}(\|\nabla g(\xi)\|^4))^{1/4} (\mathbf{E}(\|H_g(\xi))\|^4))^{1/4}}{\mathrm{Var}(F)},$$

where $H_g(\xi)$ is the Hessian matrix of g at some point ξ, and C is a universal constant.

In the Poisson framework, by plugging the Mehler formula (1.11) into the abstract bound (1.12), Last, Peccati and Schulte [LPS16] have derived the following inequality: for centred $F \in \mathscr{L}^2(\eta)$ with $\mathrm{Var}(F) = 1$, $\mathsf{d}_W(F, N) \leqslant \sum_{i=1}^{6} B_i$ where the B_i only involve the moments of $\mathsf{D}F$ and D^2F up to the order 4:

$$B_1 = 4\left[\int_{\mathbb{X}^3}\left[\mathbf{E}(\mathsf{D}_{x_1}F)^2(\mathsf{D}_{x_2}F)^2\right]^{1/2}\left[\mathbf{E}(\mathsf{D}^2_{x_1,x_3}F)^2(\mathsf{D}_{x_2,x_3}F)^2\right]^{1/2}\mu^3(dx_1,dx_2,dx_3)\right]$$

$$B_2 = \left[\int_{\mathbb{X}^3}\mathbf{E}(\mathsf{D}^2_{x_1,x_3}F)^2\mathbf{E}(\mathsf{D}^2_{x_2,x_3}F)^2\mu^3(dx_1,dx_2,dx_3)\right]^{1/2}$$

$$B_3 = \int_{\mathbb{X}}\mathbf{E}\left|\mathsf{D}_xF\right|^3\mu(dx)$$

$$B_4 = \frac{1}{2}\left[\mathbf{E}F^4\right]^{1/4}\int_{\mathbb{X}}\left[\mathbf{E}((\mathsf{D}_xF)^4)\right]^{3/4}\mu(dx)$$

$$B_5 = \left[\int_{\mathbb{X}}\mathbf{E}(\mathsf{D}_xF)^4\mu(dx)\right]^{1/2}$$

$$B_6 = \left[\int_{\mathbb{X}}6\left[\mathbf{E}(\mathsf{D}_{x_1}F)^4\right]^{1/2}\left[\mathbf{E}(\mathsf{D}^2_{x_1,x_2}F)^4\right] + 3\mathbf{E}(\mathsf{D}^2_{x_1,x_2}F)^4\mu^2(dx_1,dx_2)\right]^{1/2}$$

As anticipated, the distance between F and N is small if for distant points $x,y \in \mathbb{X}$, $\mathbf{E}(\mathsf{D}^2_{x,y}F)^4$ is small.

We will use such inequalities in Section 4 to give Central Limit Theorems with what we believe to be optimal speed of convergence for some geometric functionals. Previous ideas relied mainly upon dependency graph techniques. Penrose, Yukich and their co-authors were also able to derive central limit for stabilising geometric functionals under weaker conditions, using a CLT for martingale differences, but with no bound on the speed of convergence. See the survey [KM10, Chapter 4] and references therein.

3.1 Difference operators for binomial input

Let $\xi_n = (X_1,\dots,X_n)$ be iid variables with law μ, and F be a real functional on $\mathbf{N}^f$ the class of finite subsets of $\mathbb{X}$. For notational purpose, we instead consider F as a symmetric functional in any number of arguments, through the abuse of notation $F(\xi_n) = F(\{X_1,\dots,X_n\})$. Let $\xi_n^{\hat{i}} = (X_k)_{k\neq i}$, and introduce the difference operator by

$$\mathsf{D}_iF(\xi_n) = F(\xi_n^{\hat{i}}) - F(\xi_n).$$

We can iterate this definition by

$$\mathsf{D}^2_{i,j}F(\xi_n) = F((X_k)_{k\neq i,k\neq j}) - F(\xi_n^{\hat{i}}) - F(\xi_n^{\hat{j}}) + F(\xi_n).$$

Let $\xi_n^j = (X_1^j,\dots,X_n^j), 1 \leqslant j \leqslant 3$, be independent copies of ξ_n. We shall use the following terminology: a random vector $\zeta = (Z_1,...,Z_n)$ is a *recombination* of $\{\xi_n^1,\xi_n^2,\xi_n^3\}$, if for every $1 \leqslant i \leqslant n$ there is j such that $\mathbf{P}(Z_i = X_i^j) = 1$.

The next statement provides a bound for the normal approximation of geometric functionals that is amenable to geometric analysis, and can be heuristically regarded as the binomial counterpart to the second order Poincaré inequalities on the Poisson space (in the Kolmogorov distance), proved in [LPS16], see the introduction.

Theorem 9. *Assume that $F(\xi_n)$ is centred, and $\sigma^2 = \mathrm{Var}(F(\xi_n)) < \infty$. Let $N \sim \mathscr{N}(0,1)$. Define*

$$B_n := \sup_{(\zeta^1,\zeta^2,\zeta^3)}\mathbf{E}\left[\mathbf{1}_{\{\mathsf{D}^2_{1,2}F(\zeta^1)\neq 0\}}\mathsf{D}_1F(\zeta^2)^2\mathsf{D}_2F(\zeta^3)^2\right],$$

$$B'_n := \sup_{(\zeta^1,\zeta^2,\zeta^3,\zeta^4)}\mathbf{E}\left[\mathbf{1}_{\{\mathsf{D}^2_{1,2}F(\zeta^2)\neq 0,\mathsf{D}^2_{1,3}F(\zeta^2)\neq 0\}}\mathsf{D}_2F(\zeta^3)^2\mathsf{D}_3F(\zeta^4)^2\right],$$

where the suprema run over tuples of vectors ζ^i that are recombinations of $\{\xi_n^1, \xi_n^2, \xi_n^3\}$. Then,

$$\mathsf{d}_K(\sigma^{-1}W, N) \leqslant \varkappa \left[\frac{n^{1/2}}{\sigma^2} \left(\sqrt{nB_n} + \sqrt{n^2 B_n'} + \sqrt{\mathbf{E}\mathsf{D}_1 F(\xi_n)^4} \right) \right. \tag{3.1}$$
$$\left. + \frac{n}{\sigma^4} \sup_{A \subseteq [n]} \mathbf{E}|F(\xi_n)\mathsf{D}_1 F(\xi_n^A)^3| + \left(\frac{n}{\sigma^3} \mathbf{E}|\mathsf{D}_1 F(\xi_n)^3| \right) \right]$$

where ξ_n^A is the vector obtained from ξ_n by replacing X_i by X_i^1 for $i \in A$, and $\varkappa$ does not depend on n or on F.

In the context of Poisson stabilisation, this bound and the Poisson analogue give very similar results, see Theorem 10.

3.2 Stabilisation radius on a metric space: Poisson and binomial input

We consider the marked framework $(\mathbb{M}, \mathscr{M}, \nu)$ where $\mathbb{X} = \mathbb{X}_g \times \mathbb{M}$ for some ground space $(\mathbb{X}_g, \mathscr{X}_g, \mu_g)$. Let η_s be a Poisson measure with intensity $s\mu_g \otimes \nu$, and ξ_n a set of n iid variables in $\mathbb{X}$ with law $\mu_g \otimes \nu$, if μ_g is a probability distribution. Let $S_s, s \geqslant 1$ be score functions, i.e. measurable functions from $\mathbb{X} \times \mathbf{N}(\mathbb{X}) \to \mathbb{R}$, and the corresponding functionals

$$F_s = \sum_{x \in \eta_s} S_s(x, \eta_s), s \geqslant 1,\ F_n' = \sum_{i=1}^n S_n(x, \xi_n), n \in \mathbb{N}. \tag{3.2}$$

Assuming finite second moment, consider the renormalized versions $\tilde{F}_s = (F_s - \mathbf{E}(F_s))\mathrm{Var}(F_s)^{-1/2}$, $\tilde{F}_n' = (F_n - \mathbf{E}(F_n))\mathrm{Var}(F_n)^{-1/2}$.

Assume $\mathbb{X}_g$ is equipped with a semi-metric d such that for some $\varkappa, \gamma > 0$,

$$\limsup_{\varepsilon \to 0} \frac{\mu(B_{\mathsf{d}}(x, r+\varepsilon)) - \mu(B_{\mathsf{d}}(x, r))}{\varepsilon} \leq \varkappa \gamma r^{\gamma - 1}, \quad r \geq 0, x \in \mathbb{X}_g, \tag{3.3}$$

where $B_{\mathsf{d}}(x, r)$ is the ball with center x and radius r in the metric d.

Two examples for ground spaces $(\mathbb{X}_g, \mathscr{X}_g, \mu_g)$ and semi-metrics d satisfying the assumption (3.3) are the following:

- Let $\mathbb{X}_g$ be a full-dimensional subset of $\mathbb{R}^d$ equipped with the induced Borel σ-field $\mathscr{X}_g$ and the usual Euclidean distance d, assume that μ_g is a measure on $\mathbb{X}_g$ with a density g with respect to the Lebesgue measure, and put $\gamma := d$. Then condition (3.3) reduces to the standard assumption that g is bounded. Indeed, if $\|g\|_\infty := \sup_{x \in \mathbb{X}} |g(x)| < \infty$, then (3.3) is obviously satisfied with $\varkappa := \|g\|_\infty \varkappa_d$, where $\varkappa_d := \pi^{d/2}/\Gamma(d/2+1)$ is the volume of the d-dimensional unit ball in $\mathbb{R}^d$.

- Let $\mathbb{X}_g \subseteq \mathbb{R}^d$ be a smooth m-dimensional subset of $\mathbb{R}^d$, $m \leq d$, equipped with a semi-metric d, and a measure μ_g on $\mathbb{X}_g$ with a bounded density g with respect to the uniform surface measure Vol_m on $\mathbb{X}_g$. We assume that the Vol_{m-1} measure of the sphere $\partial(B_{\mathsf{d}}(x, r))$ is bounded by the surface area of the Euclidean sphere $\mathbb{S}^{m-1}(0, r)$ of the same radius, that is to say

$$\mathrm{Vol}_{m-1}(\partial B_{\mathsf{d}}(x, r)) \leq m \varkappa_m r^{m-1},\ x \in \mathbb{X}_g,\ r > 0. \tag{3.4}$$

When $\mathbb{X}_g$ is an m-dimensional affine space and d is the usual Euclidean metric on $\mathbb{R}^d$, (3.4) holds with equality, naturally. However (3.4) holds in more general situations. For example, by Bishop's comparison theorem (Theorem 1.2 of [SY94], along with (1.15) there), (3.4) holds for Riemannian manifolds $\mathbb{X}_g$ with non-negative Ricci curvature, with d the geodesic distance. Given the bound (3.4), one obtains (3.3) with $\varkappa = \|g\|_\infty \varkappa_m$ and $\gamma = m$. This example includes the case $\mathbb{X}_g = \mathbb{S}^m$, the unit sphere in $\mathbb{R}^{m+1}$ equipped with the geodesic distance.

To derive central limit theorems for F_s and F'_n, we impose several conditions on the scores. For $s \geq 1$ a measurable map $R_s : \mathbb{X} \times \mathbf{N}(\mathbb{X}) \to \mathbb{R}$ is called a *radius of stabilization* for S_s if for all $x \in \mathbb{X}$, $\zeta \in \mathbf{N}(\mathbb{X})$ and finite $\mathcal{A} \subset \mathbb{X}$ with $|\mathcal{A}| \leq 7$ we have

$$S_s(x, (\zeta \cup \{x\} \cup \mathcal{A}) \cap B_{\overline{\mathsf{d}}}(x, R_s(x, \zeta \cup \{x\}))) = S_s(x, \zeta \cup \{x\} \cup \mathcal{A}), \tag{3.5}$$

recall that $B_{\overline{\mathsf{d}}}(x, r) := B_{\mathsf{d}}(y, r) \times \mathbb{M}$ for $x = (y, m) \in \mathbb{X}$ and $r > 0$.

For a given point $x \in \mathbb{X}_g$ we denote by M_x the corresponding random mark, which is distributed according to ν and is independent of everything else. Say that $(S_s)_{s\geq1}$ (resp. $(S_n)_{n\in\mathbb{N}}$) are *exponentially stabilizing* if there are radii of stabilization $(R_s)_{s\geq1}$ (resp. $(R_n)_{n\in\mathbb{N}}$) and constants $C_{stab}, c_{stab}, \alpha_{stab} \in (0, \infty)$ such that, for $x \in \mathbb{X}_g$, $r \geq 0$ and $s \geq 1$,

$$\mathbf{P}(R_s((x, M_x), \eta_s \cup \{(x, M_x)\}) \geq r) \leq C_{stab} \exp(-c_{stab}(s^{1/\gamma} r)^{\alpha_{stab}}), \tag{3.6}$$

resp. for $x \in \mathbb{X}_g$, $r \geq 0$ and $n \geq 9$,

$$\mathbf{P}(R_n((x, M_x), \xi_{n-8} \cup \{(x, M_x)\}) \geq r) \leq C_{stab} \exp(-c_{stab}(n^{1/\gamma} r)^{\alpha_{stab}}), \tag{3.7}$$

where γ is the constant from (3.3).

For a finite set $\mathcal{A} \subset \mathbb{X}_g$ we denote by $(\mathcal{A}, M_\mathcal{A})$ the random set obtained by equipping each point of $\mathcal{A}$ with a random mark distributed according to ν and independent of everything else. Given $p \in [0, \infty)$, say that $(S_s)_{s\geq1}$ or $(S_n)_{n\in\mathbb{N}}$ satisfy a $(4+p)$th moment condition if there is a constant $C_p \in (0, \infty)$ such that for all $\mathcal{A} \subset \mathbb{X}$ with $|\mathcal{A}| \leq 7$,

$$\sup_{s\in[1,\infty)} \sup_{x\in\mathbb{X}_g} \mathbf{E}|S_s((x, M_x), \eta_s \cup \{(x, M_x)\} \cup (\mathcal{A}, M_\mathcal{A}))|^{4+p} \leq C_p \tag{3.8}$$

or

$$\sup_{n\in\mathbb{N}, n\geq9} \sup_{x\in\mathbb{X}_g} \mathbf{E}|S_n((x, M_x), \xi_{n-8} \cup \{(x, M_x)\} \cup (\mathcal{A}, M_\mathcal{A}))|^{4+p} \leq C_p. \tag{3.9}$$

We introduce a further notion relevant for functionals whose variances exhibit surface area order scaling. Let K be a measurable subset of $\mathbb{X}_g$ such that the map $\mathbb{X} \ni z \mapsto \mathsf{d}(z, K) := \inf_{y\in K} \mathsf{d}(z, y)$ is measurable. Here, $\mathsf{d}(z, K)$ is the distance between a point $z \in \mathbb{X}$ and K. If there is a sequence in K that is dense with respect to d, the measurability assumption is always satisfied. Moreover, we use the abbreviation $\mathsf{d}_s(\cdot, \cdot) := s^{1/\gamma}\mathsf{d}(\cdot, \cdot)$, $s \geq 1$. Say that $(S_s)_{s\geq1}$, resp. $(S_n)_{n\in\mathbb{N}}$, *decay exponentially fast with the distance to* K if there are constants $C_K, c_K, \alpha_K \in (0, \infty)$ such that for all $\mathcal{A} \subset \mathbb{X}_g$ with $|\mathcal{A}| \leq 7$ we have

$$\mathbf{P}(S_s((x, M_x), \eta_s \cup \{(x, M_x)\} \cup (\mathcal{A}, M_\mathcal{A})) \neq 0) \leq C_K \exp(-c_K \mathsf{d}_s(x, K)^{\alpha_K}) \tag{3.10}$$

for $x \in \mathbb{X}_g$ and $s \geq 1$ resp.

$$\mathbf{P}(S_n((x, M_x), \xi_{n-8} \cup \{(x, M_x)\} \cup (\mathcal{A}, M_\mathcal{A})) \neq 0) \leq C_K \exp(-c_K \mathsf{d}_n(x, K)^{\alpha_K}) \tag{3.11}$$

for $x \in \mathbb{X}_g$ and $n \geq 9$. Moreover, let $\alpha := \min\{\alpha_{stab}, \alpha_K\}$ and

$$I_{K,s} := s \int_{\mathbb{X}_g} \exp\left(- \frac{\min\{c_{stab}, c_K\} p \mathsf{d}_s(x, K)^\alpha}{36 \cdot 4^{\alpha+1}} \right) \mu_g(dx), \quad s \geq 1. \tag{3.12}$$

To prove central limit theorems, we will have to show that $\sqrt{I_{K,s}}/\mathrm{Var}F_s \to 0$ as $s \to \infty$ and $\sqrt{I_{K,n}}/\mathrm{Var}F_n' \to 0$ as $n \to \infty$, respectively. In the sequel we always assume that $I_{K,s}$ is finite. This is only a restriction for the Poisson case, where $\mu(\mathbb{X}) = \infty$ is allowed. Then, $I_{K,s} < \infty$ and (3.10) imply that the number of points of η_s with non-vanishing scores is finite almost surely. Now let us discuss some prominent choices for K. If there is a set A such that $\mu(A) < \infty$ and the scores of points outside of A vanish almost surely, we make the choice $K = A$, for which (3.10) and (3.11) are obviously satisfied with $C_K = 1$ and arbitrary $c_K, \alpha_K \in (0, \infty)$. This approach should work for functionals whose variances have volume order. If $\mathbb{X}_g$ is $\mathbb{R}^d$ or a compact convex subset of $\mathbb{R}^d$ such as the unit cube, one sometimes has to choose K to be a $(d-1)$-dimensional subset of $\mathbb{R}^d$ to ensure that $\sqrt{I_{K,s}}/\mathrm{Var}F_s$ and $\sqrt{I_{K,n}}/\mathrm{Var}F_n'$ vanish. This situation arises, for example, in statistics of convex hulls of random samples and Voronoï set approximation. Problems with surface order scaling of the variance are typically of this form.

The following general theorem provides rates of normal convergence for F_s and F_n' in terms of the Kolmogorov distance. This theorem is a consequence of general theorems from [LPS16] and [LP17] giving Malliavin-Stein bounds for functionals of Poisson and binomial point processes (see the introduction and Theorem 9 above). Throughout this paper N always denotes a standard Gaussian random variable.

Theorem 10 ([LSY19]). *(a) Assume that the score functions $(S_s)_{s\geq 1}$ are exponentially stabilizing* (3.6)*, satisfy the moment condition* (3.8) *for some $p \in (0,1]$, and decay exponentially fast with the distance to a measurable set $K \subseteq \mathbb{X}_g$, as at* (3.10)*. Then there is a constant $\tilde{C} \in (0,\infty)$ only depending on the constants in* (3.3), (3.6), (3.8) *and* (3.10) *such that*

$$\mathsf{d}_K\left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\mathrm{Var}F_s}}, N\right) \leq \tilde{C}\left(\frac{\sqrt{I_{K,s}}}{\mathrm{Var}F_s} + \frac{I_{K,s}}{(\mathrm{Var}F_s)^{3/2}} + \frac{I_{K,s}^{5/4} + I_{K,s}^{3/2}}{(\mathrm{Var}F_s)^2}\right), \quad s \geq 1. \tag{3.13}$$

(b) Assume that the score functions $(S_n)_{n\in\mathbb{N}}$ are exponentially stabilizing (3.7)*, satisfy the moment condition* (3.9) *for some $p \in (0,1]$, and decay exponentially fast with the distance to a measurable set $K \subseteq \mathbb{X}_g$, as at* (3.11)*. Let $(I_{K,n})_{n\in\mathbb{N}}$ be as in* (3.12)*. Then there is a constant $\tilde{C} \in (0,\infty)$ only depending on the constants in* (3.3), (3.7), (3.9) *and* (3.11) *such that*

$$\mathsf{d}_K\left(\frac{F_n' - \mathbf{E}F_n'}{\sqrt{\mathrm{Var}F_n'}}, N\right) \leq \tilde{C}\left(\frac{\sqrt{I_{K,n}}}{\mathrm{Var}F_n'} + \frac{I_{K,n}}{(\mathrm{Var}F_n')^{3/2}} + \frac{I_{K,n} + I_{K,n}^{3/2}}{(\mathrm{Var}F_n')^2}\right), \quad n \geq 9. \tag{3.14}$$

Notice that if $K = \mathbb{X}_g$, we have

$$I_{\mathbb{X},s} = s\mu(\mathbb{X}), \quad s \geq 1, \quad \text{and} \quad I_{\mathbb{X},n} = n\mu(\mathbb{X}), \quad n \in \mathbb{N}. \tag{3.15}$$

Assuming growth bounds on $I_{K,s}/\mathrm{Var}F_s$ and $I_{K,n}/\mathrm{Var}F_n'$, the rates (3.13) and (3.14) nicely simplify into presumably optimal rates, ready for off-the-shelf use in applications.

Corollary 11. *(a) Let the conditions of Theorem 10(a) prevail. Assume further that there is a $C \in (0,\infty)$ such that $\sup_{s\geq 1} I_{K,s}/\mathrm{Var}F_s \leq C$. Then there is a $\tilde{C}' \in (0,\infty)$ only depending on C and the constants in* (3.3), (3.6), (3.8) *and* (3.10) *such that*

$$\mathsf{d}_K\left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\mathrm{Var}F_s}}, N\right) \leq \frac{\tilde{C}'}{\sqrt{\mathrm{Var}F_s}}, \quad s \geq 1. \tag{3.16}$$

(b) Let the conditions of Theorem 10(b) prevail. If there is a $C \in (0,\infty)$ such that $\sup_{n\geq 1} I_{K,n}/\mathrm{Var}F_n' \leq C$, then there is a $\tilde{C}' \in (0,\infty)$ only depending on C and the constants in (3.3), (3.7), (3.9) *and* (3.11) *such that*

$$\mathsf{d}_K\left(\frac{F_n' - \mathbf{E}F_n'}{\sqrt{\mathrm{Var}F_n'}}, N\right) \leq \frac{\tilde{C}'}{\sqrt{\mathrm{Var}F_n'}}, \quad n \geq 9. \tag{3.17}$$

This corollary is applied in the context of the convex hull of a random sample of points in a smooth convex set in Subsection 4.5. In this case, the variance is of order $s^{\frac{d-1}{d+1}}$ ($n^{\frac{d-1}{d+1}}$ in the binomial setting), and we obtain rates of normal convergence of order $(\mathrm{Var}F_s)^{-1/2} = \Theta(s^{-(d-1)/(2(d+1))})$ (resp. $(\mathrm{Var}F_n')^{-1/2} = \Theta(n^{-(d-1)/(2(d+1))})$), which improves upon rates obtained via other methods.

In the setting $\mathbb{X}_g \subseteq \mathbb{R}^d$, our results admit further simplification, which goes as follows. For $K \subseteq \mathbb{X}_g \subseteq \mathbb{R}^d$ and $r \in (0,\infty)$, let $K_r := \{y \in \mathbb{R}^d : \mathsf{d}(y,K) \leq r\}$ denote the r-parallel set of K. Recall that the $(d-1)$-dimensional upper Minkowski content of K is given by

$$\overline{\mathcal{M}}^{d-1}(K) := \limsup_{r\to 0} \frac{\mathrm{Vol}_d(K_r)}{2r}. \tag{3.18}$$

If K is a closed $(d-1)$-rectifiable set in $\mathbb{R}^d$ (i.e., the Lipschitz image of a bounded set in $\mathbb{R}^{d-1}$), then $\overline{\mathcal{M}}^{d-1}(K)$ exists and coincides with a scalar multiple of $\mathcal{H}^{d-1}(K)$, the $(d-1)$-dimensional Hausdorff measure of K. Given an unbounded set $I \subset (0,\infty)$ and two families of real numbers $(a_i)_{i\in I}, (b_i)_{i\in I}$, we use the Landau notation $a_i = O(b_i)$ to indicate that $\limsup_{i\in I, i\to\infty} |a_i|/|b_i| < \infty$. If $b_i = O(a_i)$ we write $a_i = \Omega(b_i)$, whereas if $a_i = O(b_i)$ and $b_i = O(a_i)$ we write $a_i = \Theta(b_i)$.

Theorem 12. *Let $\mathbb{X}_g \subseteq \mathbb{R}^d$ be full-dimensional, let μ have a bounded density with respect to Lebesgue measure and let the conditions of Theorem 10 prevail with $\gamma := d$.*

(a) Let K be a full-dimensional compact subset of $\mathbb{X}_g$ with $\overline{\mathcal{M}}^{d-1}(\partial K) < \infty$. If $\mathrm{Var}F_s = \Omega(s)$, resp. $\mathrm{Var}F_n' = \Omega(n)$, then there is a constant $c \in (0,\infty)$ such that

$$\mathsf{d}_K\left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\mathrm{Var}F_s}}, N\right) \leq \frac{c}{\sqrt{s}}, \quad \textit{resp.} \quad \mathsf{d}_K\left(\frac{F_n' - \mathbf{E}F_n'}{\sqrt{\mathrm{Var}F_n'}}, N\right) \leq \frac{c}{\sqrt{n}} \tag{3.19}$$

for $s \geq 1$, resp. $n \geq 9$.

(b) Let K be a $(d-1)$-dimensional compact subset of $\mathbb{X}_g$ with $\overline{\mathcal{M}}^{d-1}(K) < \infty$. If $\mathrm{Var}F_s = \Omega(s^{(d-1)/d})$, resp. $\mathrm{Var}F_n' = \Omega(n^{(d-1)/d})$, then there is a constant $c \in (0,\infty)$ such that

$$\mathsf{d}_K\left(\frac{F_s - \mathbf{E}F_s}{\sqrt{\mathrm{Var}F_s}}, N\right) \leq \frac{c}{s^{\frac{1}{2}-\frac{1}{2d}}}, \quad \textit{resp.} \quad \mathsf{d}_K\left(\frac{F_n' - \mathbf{E}F_n'}{\sqrt{\mathrm{Var}F_n'}}, N\right) \leq \frac{c}{n^{\frac{1}{2}-\frac{1}{2d}}} \tag{3.20}$$

for $s \geq 1$, resp. $n \geq 9$.

The constants $c \in (0,\infty)$ in (a) and (b) depend on the constants in (3.3) *and* (3.6)–(3.11)*, the set K and the behavior of the variances $\mathrm{Var}F_s$ and $\mathrm{Var}F_n'$.*

Remarks. (i) *Comparing* (3.19) *with existing results.* The results at (3.19) are applicable in the setting of volume order scaling of the variances, i.e., when the variances of F_s and F_n' exhibit scaling proportional to s and n. The rate for Poisson input in (3.19) improves upon the rate given by Theorem 2.1 of [PY05] (see also Lemma 4.4 of [Pen07]), Corollary 3.1 of [BX06], and Theorem 2.3 in [PR08], which all contain extraneous logarithmic factors and which rely on dependency graph methods. The rate in (3.19) for binomial input is new, as up to now there are no explicit general rates of normal convergence for sums of stabilizing score functions S_n of binomial input.

(ii) *Comparing* (3.20) *with existing results.* The rates at (3.20) are relevant for statistics with surface area rescaling of the variances, i.e., when the variance of F_s (resp. F_n') exhibits scaling proportional to $s^{1-1/d}$ (resp. $n^{1-1/d}$), when the scores are properly renormalized. These rates both improve and extend upon the rates given in the main result (Theorem 1.3) in [Yuk15]. First, in the case of Poisson input, the rates remove the logarithmic factors present in Theorem 1.3 of [Yuk15]. Second, we obtain rates of normal convergence for binomial input, whereas [Yuk15] does not treat this situation.

(iii) *Unspecified constants.* The bounds for normal approximation all involve unspecified constants $\tilde{C}$, $\tilde{C}'$, and c. While one could explicitly determine these constants in terms of the constants in (3.3), (3.6)-(3.11) by following our proofs, we have decided for the sake of readability to not take this approach.

(iv) *Normal approximation via re-scaling.* Let η be a homogenous Poisson point process of intensity one on $\mathbb{R}^d$. Applications often involve showing normal approximation for $\sum_{x\in\eta\cap W_s}\xi(x,\eta\cap\tilde{Q}_s)$, where ξ is a stabilizing score function and $\tilde{Q}_s := [-\frac{1}{2}s^{1/d}, \frac{1}{2}s^{1/d}]^d$. To see that our main results treat this situation, it suffices to put $\mathbb{X}_g$ to be W_1, μ Lebesgue measure on $K := \tilde{Q}_1$, η_s a Poisson point process of intensity $s\mu$, and $S_s(x,\zeta) := \xi(s^{1/d}x, s^{1/d}\zeta)$. One may likewise deduce central limit theorems for $\sum_{x\in\eta\cap\tilde{Q}_s}\xi(x,\eta)$ by taking $\mathbb{X}_g$ to be $\mathbb{R}^d$, μ Lebesgue measure on $\mathbb{R}^d$, η_s a Poisson point process of intensity $s\mu$, $K := \tilde{Q}_1$, and $S_s(x,\zeta) := \xi(s^{1/d}x, s^{1/d}\zeta)$ when $x \in \tilde{Q}_1$ and zero otherwise. In this situation we have $I_{K,x} = \Theta(s)$.

(v) *Extensions to random measures.* Up to a constant factor, the rates of normal convergence in Theorem 10, Corollary 11, and Theorem 12 hold for the non-linear statistics $F_s(f) = \sum_{x\in\eta_s} f(x)S_s(x,\eta_s)$ and $F_n'(f) = \sum_{x\in\mathbb{X}_n} f(x)S_n(x,\mathbb{X}_n)$, obtained by integrating the random measures $\sum_{x\in\eta_s} S_s(x,\eta_s)\delta_x$ and $\sum_{x\in\mathbb{X}_n} S_n(x,\mathbb{X}_n)\delta_x$ with a bounded measurable test function f on $\mathbb{X}_g$. For example, if the assumptions of Theorem 12(a) are satisfied with $K = \mathbb{X}_g$, $\mathrm{Var}(F_s(f)) = \Omega(s)$, and $\mathrm{Var}(F_n'(f)) = \Omega(n)$, then there is a constant $c \in (0,\infty)$ such that

$$\mathsf{d}_K\left(\frac{F_s(f) - \mathbf{E}F_s(f)}{\sqrt{\mathrm{Var}F_s(f)}}, N\right) \leq \frac{c}{\sqrt{s}}, \quad s \geq 1, \tag{3.21}$$

and

$$\mathsf{d}_K\left(\frac{F_n'(f) - \mathbf{E}F_n'(f)}{\sqrt{\mathrm{Var}F_n'(f)}}, N\right) \leq \frac{c}{\sqrt{n}}, \quad n \geq 9. \tag{3.22}$$

Here, the constant $c \in (0,\infty)$ depends on the constants in (3.3) and (3.6)–(3.9), the set K and the behavior of the variances $\mathrm{Var}(F_s(f))$ and $\mathrm{Var}(F_n'(f))$. The rate (3.21) improves upon the main result (Theorem 2.1) of [PY05] whereas the rate (3.22) is new.

(vi) *Extensions to the Wasserstein distance.* All quantitative bounds presented in this section also hold for the Wasserstein distance (see also the discussion at the end of Section 4 in [LSY19]). Recall that the Wasserstein distance between random variables Y and Z with $\mathbf{E}|Y|, \mathbf{E}|Z| < \infty$ is given by

$$\mathsf{d}_W(Y,Z) := \sup_{h\in\mathrm{Lip}(1)} |\mathbf{E}h(Y) - \mathbf{E}h(Z)|, \tag{3.23}$$

where Lip(1) stands for the set of all functions $h : \mathbb{R} \to \mathbb{R}$ whose Lipschitz constant is at most one. Since we believe that the Kolmogorov distance d_K is more prominent than the Wasserstein distance, the applications in Section 4 are formulated only for d_K.

(vii) *Subsets without influence.* Assume that there is a measurable set $\tilde{\mathbb{X}}_g \subset \mathbb{X}_g$ such that the scores satisfy

$$S_s(x,\zeta) = \mathbf{1}_{\{x\in\tilde{\mathbb{X}}\}} S_s(x,\zeta\cap\tilde{\mathbb{X}}_g), \quad \zeta \in \mathbf{N}(\mathbb{X}), x \in \zeta, s \geq 1,$$

where $\zeta \cap \tilde{\mathbb{X}}_g$ stands for the restriction of the point configuration ζ to $\tilde{\mathbb{X}}_g$. In other words, the sum of scores $\sum_{x\in\zeta} S_s(x,\zeta)$ only depends on the points of ζ which spatial location belongs to $\tilde{\mathbb{X}}_g$. In this case our previous results are still valid if the assumptions (3.3)-(3.11) hold for all $x \in \tilde{\mathbb{X}}_g$.

(viii) *Null sets.* In our assumptions (3.3)-(3.11) we require, for simplicity, that some inequalities are satisfied for all $x \in \mathbb{X}$. In case that these only hold for μ-a.e. $x \in \mathbb{X}$, our results are still true. This also applies to comment (vii).

3.3 Functionals without a stabilisation radius

This section is about functionals who don't admit a radius of stabilisation per se, but satisfy stabilisation in a weaker sense. Very distant points will have an influence in the sense that if one point far away from 0 is removed, the contributions of points around 0 are modified, but this modification should be very small. Second-order Poincare inequalities are actually powerful because they can treat such functionals. Note that the underlying space is still under the form $\mathbb{R}^d \times \mathbb{M}$ as in Section 2.2, but $\mathbb{X}_g = \mathbb{R}^d$ in all the section. The notation of Section 2.2 should prevail. For simplicity, we don't state the result for the truncated input, i.e. with $F'_{\tilde{W}}$ instead of $F_{\tilde{W}}$, but refer the reader to [Lac19].

Theorem 13. *Let $W \subset \mathbb{Z}^d$ bounded. Let $F_{\tilde{W}}$ as defined in* (2.3)*, and let $M, M' \sim \nu$ independent. Assume that for some $C_0 > 0$, $\alpha > 5d/2$, Assumption 6 holds. Then, $\sigma^2 := \mathrm{Var}(F_{\tilde{W}}) < \infty$. If furthermore $\sigma > 0$, with $\tilde{F}_{\tilde{W}} = \sigma^{-1}(F_{\tilde{W}} - \mathbf{E}F_{\tilde{W}})$,*

$$\mathsf{d}_W(\tilde{F}_{\tilde{W}}, N) \leqslant \varkappa \left(C_0^2\sigma^{-2}\sqrt{|W|} + C_0^3\sigma^{-3}|W|\right)\left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^{2(\alpha/d-2)}\right). \tag{3.24}$$

Let $v := \sup_W (F_{\tilde{W}} - \mathbf{E}F_{\tilde{W}})^4 |W|^{-2} \in \mathbb{R}_+ \cup \{\infty\}$, then

$$\mathsf{d}_K(\tilde{F}_W, N) \leqslant \varkappa \left(C_0^2\sigma^{-2}\sqrt{|W|} + C_0^3\sigma^{-3}|W| + v^{1/4}C_0^3\sigma^{-4}|W|^{3/2}\right)\left(1 + \left(\frac{|\partial_{\mathbb{Z}^d} W|}{|W|}\right)^{a}\right). \tag{3.25}$$

Recall that in virtue of Theorem 8, if furthermore the non-degeneracy Assumption 7 is satisfied, (2.8) holds and in particular we have $\sigma \geqslant \varkappa\sigma_\infty|W| > 0$ in the result above. Also, if $\alpha > 2d, v < \infty$

Chapter 4

Applications

By appropriately choosing the measure space $(\mathbb{X}, \mathscr{X}, \mu)$, and either the functional F_0 (Sections 2.2,3.3) or the scores $(S_s)_{s\geq 1}$ and $(S_n)_{n\in\mathbf{N}}$, and the set $K \subset \mathbb{X}_g$ (Section 3.2), we may use Theorem 8,Theorem 10, Corollary 11, Theorem 12 and Theorem 13 to deduce presumably optimal rates of normal convergence for statistics in geometric probability. For example, in the setting $\mathbb{X} = \mathbb{R}^d$, we expect that all of the statistics F_s and F'_n described in [BY05; Pen07; PY01; PY03; PY05] consist of sums of scores S_s and S_n satisfying the conditions of Theorem 12, showing that the statistics in these papers enjoy rates of normal convergence (in the Kolmogorov distance) given by the reciprocal of the standard deviation of F_s and F'_n, respectively. Previously, the rates in these papers either contained extraneous logarithmic factors, as in the case of Poisson input, or the rates were sometimes non-existent, as in the case of binomial input. In the following we do this in detail for some prominent statistics featuring in the stochastic geometry literature, including the k-face and intrinsic volume functionals of convex hulls of random samples. Our selection of statistics is illustrative rather than exhaustive and is intended to demonstrate the wide applicability of Theorem 10 and the relative simplicity of Corollary 11 and Theorem 12. In some instances the rates of convergence are subject to variance lower bounds, a separate problem addressed in Chapter 2.

We believe that one could use our approach to also deduce presumably optimal rates of normal convergence for statistics of random sequential packing problems as in [SPY07], set approximation via Delaunay triangulations as in [JY11], generalized spacings as in [BPY09], and general proximity graphs as in [GJL18].

We start this section by presenting application to the excursion sets of random Poisson shot noise fields.

4.1 Shot noise excursions

Let $(\mathbb{M}, \mathscr{M}, \nu)$ be some probability space and $\{g_m; m \in \mathbb{M}\}$ be a set of measurable functions $\mathbb{R}^d \to \mathbb{R}$ not containing the function $g \equiv 0$. For $\zeta \in \mathbf{N}(\mathbb{X})$, introduce the shot noise field

$$\mathsf{f}_\zeta(y) = \sum_{(x,m)\in\zeta} g_m(y-x), y \in \mathbb{R}^d. \tag{4.1}$$

If η is a marked Poisson process with intensity measure $\mathcal{L}^d \otimes \nu$ on $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$, f_η is the homogeneous Poisson shot noise field with impulse distribution ν. We give later conditions under which this field is well defined. A shot noise field is the result of random functions translated at random locations in the space. It has been introduced by Campbell to model thermionic noise [Cam09], and has been used since then under different names in many fields such as pharmacology, mathematical morphology [Lan02, Section 14.1], image analysis [Gal+12], or telecommunication networks [BB83; BB10]. Biermé and Desolneux [BD12; BD16a; BD16b] have computed the mean values for some geometric properties of excursions. More generally, the activity about asymptotic properties of random fields excursions has recently increased, with the notable recent contribution of Estrade and Léon [EL16], who derived a central limit theorem for the Euler characteristic of excursions of stationary Euclidean Gaussian fields. Bulinski, Spodarev and Timmerman [BST12] give general conditions for asymptotic normality of the excursion volume for quasi-associated random fields. Their results apply to shot-noise fields, under conditions of non-negativity and uniformly bounded marginal density. We give here the asymptotic variance and central limit theorems for volume and perimeter of excursions under weak assumptions on the density. Our results allow to treat fields with singularities, such as those observed in astrophysics or telecommunications, see [BB83].

Given some threshold $u \in \mathbb{R}$, we consider the excursion set $\{\mathsf{f}_\zeta \geqslant u\} = \{x \in \mathbb{R}^d : \mathsf{f}_\zeta(x) \geqslant u\}$ and the functionals $\zeta \mapsto \mathcal{L}^d(\{\mathsf{f}_\zeta \geqslant u\} \cap \tilde{W})$, $\zeta \mapsto \mathrm{Per}(\{\mathsf{f}_\zeta \geqslant u\}; \tilde{W})$, where for $A, B \subset \mathbb{R}^d; \mathrm{Per}(A; B)$ denotes the amount of perimeter of A contained in B in the variational sense, see Section 4.1.2. The total curvature, related to the Euler characteristic is also studied in Section 4.1.3 for a specific form of the kernels.

The results apply mainly to some smooth shot noise fields, but also to processes that can be written under the form

$$\mathsf{f}_\zeta(x) = \sum_{i \geqslant 1} L_i \mathbf{1}_{\{x - x_i \in A_i\}}, x \in \mathbb{R}^d, \tag{4.2}$$

where the $(L_i, A_i), i \geqslant 1$ are iid couples of $\mathbb{R} \times \mathscr{B}_d$, endowed with the product σ-algebra and a probability measure, see Section 4.1.3. Such models are called *dilution functions* or *random token models* in mathematical morphology, see for instance [Lan02, Section 14.1], where they are used to simulate random functions with a prescribed covariance.

To the best of our knowledge, the results about the perimeter or the Euler characteristic are the first of their kind for shot noise models, and the results about the volume improve existing results, see the beginning of Section 4.1.1 for more details. Let the notation of the introduction prevail. For the process f_η (see (4.1)) to be well defined, assume throughout the section that for some $\tau > 0$,

$$\int_{\mathbb{M}} \int_{B(0,\tau)^c} |g_m(x)| dx \nu(dm) < \infty, \tag{4.3}$$

and let $\mathbf{N}_\nu$ be the class of locally finite ζ such that $\sum_{(x,m) \in \zeta} |g_m(x)| \in \mathbb{R} \cup \{\pm\infty\}$ is well defined for $\mathcal{L}$-a.a. $x \in \mathbb{R}^d$. The fact that $\eta \in \mathbf{N}_\nu$ a.s. follows from the Campbell-Mecke formula.

We study in this section the behaviour of functionals of the excursion set $\{\mathsf{f}_\eta \geqslant u\}, u \geqslant 0$. We use the general framework of random measurable sets. A *random measurable set* is a random variable taking values in the space $\mathscr{B}_d$ of measurable subsets of $\mathbb{R}^d$, endowed with the Borel σ-algebra $\mathscr{B}(\mathscr{B}_d)$ induced by the local convergence in measure, see Section 9.2. Regarding the more familiar setup of random closed sets, in virtue of Theorem 93, a random measurable set which realisations are a.s. closed can be assimilated to a random closed set.

4.1.1 Volume of excursions

For $u \in \mathbb{R}$ fixed, $W \subset \mathbb{Z}^d, \zeta \in \mathbf{N}_\nu$, define

$$F_W(\zeta) = \mathcal{L}^d(\{\mathsf{f}_{\zeta \cap \tilde{W}} \geqslant u\} \cap \tilde{W}), \quad F'_W(\zeta) = \mathcal{L}^d(\{\mathsf{f}_\zeta \geqslant u\} \cap \tilde{W}).$$

A central limit theorem for the volume of non-negative shot noise excursions has been derived in [BST12], under the assumption that $\mathsf{f}_\eta(0)$ has a uniformly bounded density and $\int |g_m(x)|\nu(dm)$ decreases sufficiently fast as $\|x\| \to \infty$, using the associativity properties of non-negative shot-noise fields. In some specific cases, the bounded density can be checked manually with computations involving the Fourier transform. In this section, we refine this result in several ways:

- A general model of random function is treated, it can in particular take negative values, allowing for compensation mechanisms (see [Lan02]). For $u > 0$, to avoid trivial cases we assume

$$\nu(\{m \in \mathbb{M} : g_m \geqslant 0\}) \neq 0. \tag{4.4}$$

- The precise variance asymptotics are derived.
- Weaker conditions are required for the results to hold, in particular bounded density is not needed.
- The likely optimal rate of convergence in Kolmogorov distance towards the normal is given.
- Boundary effects under finite input are considered, in the sense that only points falling in a bounded window (growing to infinity) contribute to the field. The case of infinite input is also treated.

We give examples of fields with no marginal density to which the results apply, such as sums of indicator functions, or kernels with a singularity in 0. Controlling the density of shot-noise fields is in general crucial for deriving results on fixed-level excursions. The case of indicator kernels is treated in Section 4.1.3.

Assumption 14. *Let f_η be of the form*

$$\mathsf{f}_\eta(x) = \sum_{y \in \eta} g(\|y - x\|), x \in \mathbb{R}^d, \tag{4.5}$$

with g non-increasing $\mathbb{R}_+ \to (0, \infty)$ such that $|g(x)| \leqslant c\|x\|^{-\lambda}, \|x\| \geqslant 1$ for some $\lambda > 11d, c > 0$. Assume that there is $\varepsilon > 0, c > 0$ such that

$$\int_0^r \frac{\rho^{-2} \wedge \rho^{2(d-1)}}{-g'(\rho)} d\rho \leqslant c \exp(cr^{d-\varepsilon}), r > 0. \tag{4.6}$$

[Lac19, Lemma 4.2] yields that if f_η satisfies this assumption, we can somehow control its density: for $a \in (0,1)$ there is $c_a > 0$ such that

$$\sup_{v \in \mathbb{R}, \delta > 0} \mathbf{P}(\mathsf{f}_\eta(0) \in (v - \delta, v + \delta)) \leqslant c_a \delta^a. \tag{4.7}$$

This result might be of independent interest. Here are examples of functions fulfilling Assumption 14 (and hence satisfying (4.7)), note that nothing prevents g from having a singularity in 0.

Example 15. Theorem 16 below applies in any dimension to $g(\rho) = C\rho^{-\nu}\mathbf{1}_{\{\rho \leqslant 1\}} + g_1(\rho)\mathbf{1}_{\{\rho > 1\}}, \rho > 0$ and $g_1(\rho)$ is for instance of the form $\exp(-a\rho^\gamma)$ or $\rho^{-\lambda}$, with $a, \nu > 0, \lambda > 11d, \gamma < d, C > 0$. Such fields don't necessarily have a finite first-order moment, and are used for instance in [BB83] to approximate stable fields, or for modeling telecommunication networks.

To give results in the case where boundary effects are considered, we need an additional hypothesis on the geometry of the underlying family of windows $\mathscr{W} = \{W_n; n \geqslant 1\}$. For $\theta > 0$, let $\mathcal{C}_\theta$ be the family of cones $C \subset \mathbb{R}^d$ with apex 0 and aperture θ, i.e. such that $\mathcal{H}^{d-1}(C \cap \mathbb{S}^{d-1}) \geqslant \theta$. Let $\mathcal{C}_{\theta,R} = \{C \cap B(0,R) : C \in \mathcal{C}_\theta\}$ for $R \geqslant 0$. Say that $\mathscr{W}$ has aperture $\theta > 0$ if for all $W \in \mathscr{W}$ with diameter $r > 0$, W has *aperture* θ : for $x \in \tilde{W}$, there is $C \in \mathcal{C}_{\theta,\ln(r)^{1/2d}}$ such that $(x + C) \subset \tilde{W}$.

Theorem 16. *Let $u > 0$. Let $G_W = F'_W$, or $G_W = F_W$ if $\mathscr{W}$ is assumed to have aperture $\theta > 0$. Assume that Assumption 14 holds. Then as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$, $\mathrm{Var}(G_W) \sim \sigma_0^2 |W|$, $(G_W - \mathbf{E}G_W)(\sigma_0\sqrt{|W|})^{-1}$ satisfies the result of Theorems 8,13 with*

$$\sigma_\infty^2 = \int_{\mathbb{R}^d} \left[\mathbf{P}(\mathsf{f}_\eta(0) \geqslant u, \mathsf{f}_\eta(x) \geqslant u) - \mathbf{P}(\mathsf{f}_\eta(0) \geqslant u)^2\right] dx > 0. \tag{4.8}$$

In particular G_W follows a central limit theorem with convergence rate (3.24) *in Kolmogorov distance.*

This result requires f to be under the form (4.5) mainly because of the density estimates implied by Assumption 14, but under general density assumptions, it could apply to more general models of the form (4.1).

4.1.2 Perimeter

We use in this section the variational definition of perimeter, following Ambrosio, Fusco and Pallara [AFP00]. Define the *perimeter* of a measurable set $A \subset \mathbb{R}^d$ within $U \subset \mathbb{R}^d$ as the total variation of its indicator function

$$\mathrm{Per}(A; U) := \sup_{\varphi \in \mathscr{C}_c^1(U,\mathbb{R}^d) : \|\varphi\|_\infty \leqslant 1} \int_{\mathbb{R}^d} \mathbf{1}_A(x) \mathrm{div}\varphi(x) dx,$$

where $\mathscr{C}_c^1(U, \mathbb{R}^d)$ is the set of continuously differentiable functions with compact support in U. Note that for regular sets, such as $\mathscr{C}^1$ manifolds, or convex sets with non-empty interior, this notion meets the classical notion of $(d-1)$-dimensional Hausdorff surface measure [AFP00, Exercise 3.10], even though the term *perimeter* is traditionally used for 2-dimensional objects. It is a possibly infinite quantity, that might also have counterintuitive features for pathological sets ([AFP00, Example 3.53]). The main difference with the traditional perimeter is that the variational one obviously cannot detect the points of the boundary whose neighborhoods don't charge the volume of the set, such as in line segments for instance. See Section 9.2.2 for a more detailed account of the variational perimeter.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and level $u \in \mathbb{R}$, the perimeter of the excursion $\mathrm{Per}(\{f \geqslant u\}; U)$ within U is a well-defined quantity. To be able to compute it efficiently, we must make additional assumptions on the regularity of f. Following [BD16b], we assume that f belongs to the space $BV(U)$ of functions with bounded variations, i.e. $f \in \mathscr{L}^1(U)$ and its variation above U is finite:

$$V(f, U) := \sup_{\varphi \in \mathcal{C}_c^1(U,\mathbb{R}^d) : \|\varphi\| \leqslant 1} \int_U f(x) \mathrm{div}\varphi(x) dx < \infty.$$

The original (equivalent) definition states that $f \in \mathscr{L}^1(U)$ is in $BV(U)$ if and only if the following holds ([AFP00, Proposition 3.6]): there exists signed Radon measures $\mathsf{D}_i f$ on $U, 1 \leqslant i \leqslant d$, called *directional derivatives* of f, such that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_U f(x) \mathrm{div}\varphi(x) dx = -\sum_{i=1}^d \int_U \varphi_i(x) \mathsf{D}_i f(dx).$$

Then there is a finite Radon measure $\|\mathsf{D}f\|$ on U, called *total variation measure*, and a $\mathbb{S}^{d-1}$-valued function $\nu_f(x), x \in U$, such that $\sum_i \mathsf{D}_i f = \|\mathsf{D}f\|\nu_f$. According to the Radon-Nikodym theorem, the total variation can be decomposed as

$$\|\mathsf{D}f\| = \nabla f \mathcal{L}^d + \mathsf{D}^j f + \mathsf{D}^c f \tag{4.9}$$

where ∇f is the density of the continuous part of $\|\mathsf{D}f\|$ with respect to $\mathcal{L}^d$, $\mathsf{D}^c f + \mathsf{D}^j f$ is the singular part of $\|\mathsf{D}f\|$ with respect to Lebesgue measure, decomposed in the *Cantor part* $\mathsf{D}^c f$, and the jump part $\mathsf{D}^j f$, that we specify below, following [AFP00, Section 3.7].

For $x \in U$, denote by H_x the affine hyperplane containing x with outer normal vector $\nu_f(x)$. For $r > 0$, denote by $B^+(x,r)$ and $B^-(x,r)$ the two components of $B(x,r) \setminus H_x$, with $\nu_f(x)$ pointing towards $B^+(x,r)$. Say that x is a *regular point* if there are two values $f^+(x) \geqslant f^-(x)$ such that

$$\lim_{r\to 0} r^{-d} \int\limits_{B^+(x,r)} |f^+(x) - f(y)|dy = \lim_{r\to 0} r^{-d} \int\limits_{B^-(x,r)} |f(y) - f^-(x)|dy = 0. \tag{4.10}$$

It turns out that the set of non-regular points has $\mathcal{H}^{d-1}$-measure 0 ([AFP00, Th. 3.77]), and the set J_f of points where $f^+(x) > f^-(x)$, called *jump points*, has Lebesgue measure 0 ([AFP00, Th. 3.83]). Then, the jump measure of f is represented by

$$\mathsf{D}^j f(dx) = \mathbf{1}_{\{x\in J_f\}}(f^+(x) - f^-(x))\mathcal{H}^{d-1}(dx),$$

where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure.

In the classical case where f is continuously differentiable on U, $\mathsf{D}f = \nabla f \mathcal{L}^d$, $\nu_f(x) = \|\nabla f(x)\|^{-1}\nabla f(x)$ (and takes an irrelevant arbitrary value if $\nabla f(x) = 0$), and $V(f;U) = \int_U \|\nabla f(x)\|dx$. If $f = \mathbf{1}_{\{A\}}$ for some $\mathscr{C}^1$ compact manifold A, $\nu_f(x)$ is the outer normal to A for $x \in \partial A$, $\nabla f = 0, \mathsf{D}^c f = 0$, and $\mathsf{D}^j f = \mathbf{1}_{\{\partial A\}}\mathcal{H}^{d-1}$.

Denote by $SBV(U)$ the functions $f \in BV(U)$ such that $\mathsf{D}^c f = 0$. Assume here that for $m \in \mathbb{M}, g_m \in SBV(\mathbb{R}^d)$, and that

$$\int_{\mathbb{M}} \left[\int_{\mathbb{R}^d} (|g_m(t)| + \|\nabla g_m(t)\|)dt + \int_{J_{g_m}} |g_m^+(t) - g_m^-(t)|\mathcal{H}^{d-1}(dt) \right] \nu(dm) < \infty.$$

Let $\mathbf{N}'_\nu$ be the class of configurations $\zeta \in \mathbf{N}_\nu$ such that the corresponding shot noise field f_ζ is of class $SBV(U)$ on every bounded set U, finite a.e. on $\mathbb{R}^d$, its gradient density defined by (4.9) is a vector-valued shot-noise field, defined a.s. and $\mathcal{L}^d$-a.e. by

$$\nabla \mathsf{f}_\zeta(t) = \sum_{(x,m)\in\zeta} \nabla g_m(t-x),$$

and its jump set J_f is the union of the translates of the impulse jump sets: $J_f = \cup_{(x,m)\in\zeta}(x + J_{g_m})$, and the jumps of f are

$$\mathsf{f}_\zeta^+(y) - \mathsf{f}_\zeta^-(y) = \sum_{(x,m)\in\zeta} \mathbf{1}_{\{y\in x+J_{g_m}\}}(g_m^+(y-x) - g_m^-(y-x)), y \in J_f.$$

[BD16b, Theorem 2] and the previous assumption yield that $\eta \in \mathbf{N}'_\nu$ a.s.. Let h be a *test function*, i.e. a function $h : \mathbb{R} \to \mathbb{R}$ of class $\mathscr{C}^1$ with compact support. Let H be a primitive function of h. Biermé and Desolneux [BD16b, Theorem 1] give for $W \subset \mathbb{Z}^d, \zeta \in \mathbf{N}$,

$$F_W^{h,Per}(\zeta) := \int_{\mathbb{R}} h(u)\mathrm{Per}(\{\mathsf{f}_\zeta \geqslant u\}; \tilde{W})du = F_W^{h,cont}(\zeta) + F_W^{h,jump}(\zeta),$$

where

$$F_W^{h,cont}(\zeta) = \int_{\tilde{W}} h(\mathsf{f}_\zeta(x))\|\nabla \mathsf{f}_\zeta(x)\|dx,$$
$$F_W^{h,jump}(\zeta) = \int_{J_f\cap\tilde{W}} (H(\mathsf{f}_\zeta^+(x)) - H(\mathsf{f}_\zeta^-(x)))\mathcal{H}^{d-1}(dx).$$

Their expectations under η are computed in [BD16b, Section 3] :

$$\mathbf{E}[F_W^{h,cont}(\eta)] = |W|\mathbf{E}\left[h(\mathsf{f}_\eta(0))\|\nabla \mathsf{f}_\eta(0)\|\right]$$
$$\mathbf{E}[F_W^{h,jump}(\eta)] = |W|\int_{\mathbb{M}}\int_{J_{g_m}}\left(\int_{g_m^-(y)}^{g_m^+(y)}\mathbf{E}[h(s+\mathsf{f}_\eta(0))]ds\right)\mathcal{H}^{d-1}(dy)\nu(dm).$$

Let us now give their second order behaviour. It is difficult to give sharp necessary conditions for non-degeneracy of the variance if the function h changes signs, so we treat the case $h \geqslant 0$, but the result can likely be extended in that direction.

Theorem 17. *Let $\mathscr{W} = \{W_n; n \geqslant 1\}$ satisfying (2.5). Assume that (4.4) holds and that $\mathbf{P}(F_W^{h,Per}(\eta) \neq F_W^{h,Per}(\emptyset)) > 0$ for some $W \subset \mathbb{Z}^d$. Assume that for some $\alpha > 5d/2, c > 0$,*

$$(\mathbf{E}|g_M(x)|^4)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}, \tag{4.11}$$
$$(\mathbf{E}\|\nabla g_M(x)\|^4)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}, \tag{4.12}$$
$$\left(\mathbf{E}\left[\int\limits_{J_{g_M}\cap(x+[0,1)^d)}(1\vee|g_M^+(t)-g_M^-(t)|)\mathcal{H}^{d-1}(dt)\right]^4\right)^{1/4} \leqslant c(1+\|x\|)^{-d-\alpha}. \tag{4.13}$$

Then the conclusions of Theorems 8,13 hold for $F_0 := F_{\{0\}}^{h,\mathrm{Per}}$. In particular, $F_W^{h,\mathrm{Per}}$ has a variance proportional to $|W|$ and follows a CLT as $|\partial_{\mathbb{Z}^d}W|/|W| \to 0$.

Example 18. Assume $\mathbb{M} = \mathbb{R}$ is endowed with a probability measure ν with finite 4-th moment. Let the field be a function of the form

$$\mathsf{f}_\zeta(x) = \sum_{(y,m)\in\zeta} mg(\|x-y\|)$$

with $g \in SBV(\mathbb{R})$. Conditions (4.11) and (4.12) hold if $|g(r)| \leqslant C(1+r)^{-d-\alpha}$ and $|g'(r)| \leqslant C(1+r)^{-d-\alpha}, r > 0$. Then (4.13) holds if J_g is countable and for some $C > 0$, for every $r > 0$

$$\sum_{t\in J_g\cap[r,r+1)}(1\vee|g^+(t)-g^-(t)|) \leqslant C(1+r)^{-d-\alpha}.$$

4.1.3 Fixed level perimeter and Euler characteristic

Let $\mathscr{B}$ be a subset of $\mathscr{B}_d$, and let the marks space be $\mathbb{M} = (\mathbb{R}\setminus\{0\})\times\mathscr{B}$, endowed with the product σ-algebra and some probability measure ν. This section is restricted to shot-noise fields of the form

$$\mathsf{f}_\zeta(x) = \sum_{(y,(L,A))\in\zeta} L\mathbf{1}_{\{x-y\in A\}}, \zeta \subset \mathbb{R}^d\times\mathbb{M}, x\in\mathbb{R}^d. \tag{4.14}$$

Such fields are used in image analysis [BD16a; BD16b], or in mathematical morphology [Lan02], sometimes with $L = const.$, and their marginals might not have a density. The article [Bie+18] uses the asymptotic normality result below for the Euler characteristic when $\mathscr{B}$ is the class of closed discs in $\mathbb{R}^2$ (Example 21).

The current framework allows to give general results for a fixed level $u \in \mathbb{R}$, for a large class of additive functionals, including the perimeter or the total curvature, related to the Euler characteristic. For the latter, the main difficulty is to properly define it properly on a typical excursion of the shot noise field, as it is obtained by locally adding and removing sets from $\mathscr{B}$. The general result only involves the marginal distribution $\nu_{\mathscr{B}}(\cdot) := \nu(\mathbb{R} \times \cdot)$.

We call $\mathscr{B}'$ the class of excursion sets generated by shot noise fields of the form (4.14) where all but finitely many points of ζ in general position have been removed. Formally, given a measurable subclass $\mathscr{B}' \subset \mathscr{B}_d$, a function $V : \mathscr{B}' \to \mathbb{R}$ such that $V(A)$ only depends on $A \cap \tilde{Q}_1$, and a function $|V| : \mathscr{B} \to (0, \infty)$, say that $(\mathscr{B}, \mathscr{B}', V, |V|)$ is *admissible* if for $A_1, \ldots, A_q \in \mathscr{B}$, for a.a. $y_1, \ldots, y_q \in \mathbb{R}^d$, any set A obtained by sequentially removing, adding or intersecting the $A_i + y_i, i = 1, \ldots, q$, belongs to $\mathscr{B}'$, and $|V(A)| \leqslant \sum_{i=1}^q |V|(A_i)$. We consider below the functionals, for $W \subset \mathbb{Z}^d, \zeta \in \mathbf{N}$,

$$F_W(\zeta) = \sum_{k \in W} V(\{\mathsf{f}_{\zeta \cap \tilde{W}} \geqslant u\} - k), \quad F'_W(\zeta) = \sum_{k \in W} V(\{\mathsf{f}_{\zeta} \geqslant u\} - k).$$

Example 19 (Volume)**.** The simplest example is the class $\mathscr{B} = \mathscr{B}_d$ of measurable subsets of $\mathbb{R}^d$, endowed with Lebesgue measure $V(A) = \mathcal{L}^d(A \cap \tilde{Q}_1)$. We have $F_W(\eta) := \mathcal{L}^d(\{\mathsf{f}_{\eta \cap \tilde{W}} \geqslant u\} \cap \tilde{W})$ a.s.. This example has been treated in a different framework at Section 4.1.1.

Example 20 (Perimeter)**.** Let $\mathscr{B}$ be the class of $A \in \mathscr{B}_d$ such that $\mathcal{H}^{d-1}(\partial A) < \infty$. Define $V(A) = \mathcal{H}^{d-1}(\partial A \cap \tilde{Q}_1)$, we prove at [Lac19][Section 4.3] that $F_W(\eta) = \mathcal{H}^{d-1}(\{\mathsf{f}_{\eta \cap \tilde{W}} \geqslant u\} \cap \tilde{W})$ a.s.. Assume for the moment condition that $\int_{\mathscr{B}} \mathcal{H}^{d-1}(\partial A)^8 \nu_{\mathscr{B}}(dA) < \infty$.

Example 21 (Total curvature)**.** Let $d = 2$, $\mathscr{B}$ be the class of non-trivial closed discs of $\mathbb{R}^2$. A set $A \subset \mathbb{R}^2$ is an *elementary set* in the terminology of Biermé & Desolneux [BD16a] if ∂A can be decomposed as a finite union of $\mathscr{C}^2$ open curves $C_j, j = 1, \ldots, p$ with respective constant curvatures $\varkappa_j > 0$, separated by corners $x_i \in \partial A, i = 1, \ldots, q$, (with $0 \leqslant q \leqslant p$) with angle $\alpha(x_i, A) \in (-\pi, \pi)$. The *total curvature* of A within some open set U is defined by

$$TC(A; U) := \sum_{j=1}^p \varkappa_j \mathcal{H}^1(C_j \cap U) + \sum_{i=1}^q \mathbf{1}_{\{x_i \in U\}} \alpha(x_i, A).$$

Therefore we define $V(A) = TC(A; \tilde{Q}_1)$. Via the Gauss-Bonnet theorem, for $W \subset \mathbb{Z}^d$, $TC(A; \mathrm{int}(\tilde{W}))$ is strongly related to the Euler characteristic of $A \cap \tilde{W}$, in the sense that they coincide if $A \subset \mathrm{int}(\tilde{W})$, and otherwise they only differ by boundary terms, see [BD16a]. We have $F_W(\eta) = TC(\{\mathsf{f}_{\eta \cap \tilde{W}} \geqslant u\}; \mathrm{int}(\tilde{W}))$ a.s. (see [Lac19][Section 4.3]). Assume also that the typical radius has a finite moment of order $8d$.

Proposition 22. *In the three previous examples, assume that for some $\lambda > 28d, C > 0$,*

$$\nu_{\mathscr{B}}(\{A : (x + A) \cap \tilde{Q}_1 \neq \emptyset\}) \leqslant C(1 + \|x\|)^{-\lambda}, x \in \mathbb{R}^d,$$

and that $\mathbf{P}(\mathsf{f}_\eta(0) \geqslant cu) \notin \{0, 1\}$ for some $c > 0$. Then the functionals F_W, F'_W satisfy the conclusions of theorems 8 and 13, in particular, they have variance of volume order and undergo a central limit theorem as $|\partial_{\mathbb{Z}^d} W|/|W| \to 0$.

Remark that nothing prevents the typical grain of $\mathscr{B}$ to be unbounded with positive $\nu_{\mathscr{B}}$-probability.

With a similar route, the previous example can likely be generalised to more general classes of sets $\mathscr{B}$ in higher dimensions, such as the polyconvex ring, provided one can estimate properly the curvature or the Euler characteristic on sets from $\mathscr{B}'$.

4.2 Nearest neighbors graphs and statistics of high-dimensional data sets

a. *Total edge length of nearest neighbors graphs.* Let $(\mathbb{X}, \mathscr{X}, \mu)$ be equipped with a semi-metric d such that (3.3) is satisfied for some γ and $\varkappa$. We equip $\mathbb{X}$ with a fixed linear order, which is possible by the well-ordering principle. Given $\zeta \in \mathbf{N}$, $k \in \mathbb{N}$, and $x \in \zeta$, let $V_k(x, \zeta)$ be the set of k nearest neighbors of x, i.e., the k closest points of x in $\zeta \setminus \{x\}$. In case that that these k points are not unique, we break the tie via the fixed linear order on $\mathbb{X}$. The (undirected) nearest neighbor graph $NG_1(\zeta)$ is the graph with vertex set ζ obtained by including an edge $\{x, y\}$ if $y \in V_1(x, \zeta)$ and/or $x \in V_1(y, \zeta)$. More generally, the (undirected) k-nearest neighbors graph $NG_k(\zeta)$ is the graph with vertex set ζ obtained by including an edge $\{x, y\}$ if $y \in V_k(x, \zeta)$ and/or $x \in V_k(y, \zeta)$. For all $q \geq 0$ define

$$S^{(q)}(x, \zeta) := \sum_{y \in V_k(x, \zeta)} \rho^{(q)}(x, y), \tag{4.15}$$

where $\rho^{(q)}(x, y) := \mathsf{d}(x, y)^q/2$ if x and y are mutual k-nearest neighbors, i.e., $x \in V_k(y, \zeta)$ and $y \in V_k(x, \zeta)$, and otherwise $\rho^{(q)}(x, y) := \mathsf{d}(x, y)^q$. The total edge length of the undirected k-nearest neighbors graph on $\mathbb{X}$ with qth power weighted edges is

$$F^{(q)}_{NG_k}(\zeta) = \sum_{x \in \mathbb{X}} S^{(q)}(x, \zeta).$$

As usual η_s is a Poisson point process on $\mathbb{X}$ with intensity measure $s\mu$ and ξ_n is a binomial point process of n points in $\mathbb{X}$ distributed according to μ. We assume in the following that $(\mathbb{X}, \mathscr{X}, \mu)$ satisfies (3.3) and

$$\inf_{x \in \mathbb{X}} \mu(B(x, r)) \geq c r^\gamma, \quad r \in [0, \mathrm{Diam}(\mathbb{X})], \tag{4.16}$$

where γ is the constant from (3.3), $\mathrm{Diam}(\mathbb{X})$ stands for the diameter of $\mathbb{X}$ and $c > 0$.

Theorem 23. *If $q \geq 0$ and $\mathrm{Var}(F^{(q)}_{NG_k}(\eta_s)) = \Omega(s^{1-2q/\gamma})$, then there is a $\tilde{C} \in (0, \infty)$ such that*

$$\mathsf{d}_K\left(\frac{F^{(q)}_{NG_k}(\eta_s) - \mathbf{E}F^{(q)}_{NG_k}(\eta_s)}{\sqrt{\mathrm{Var}F^{(q)}_{NG_k}(\eta_s)}}, N\right) \leq \frac{\tilde{C}}{\sqrt{s}}, \quad s \geq 1, \tag{4.17}$$

whereas if $\mathrm{Var}(F^{(q)}_{NG_k}(\xi_n)) = \Omega(n^{1-2q/\gamma})$, then

$$\mathsf{d}_K\left(\frac{F^{(q)}_{NG_k}(\xi_n) - \mathbf{E}F^{(q)}_{NG_k}(\xi_n)}{\sqrt{\mathrm{Var}F^{(q)}_{NG_k}(\xi_n)}}, N\right) \leq \frac{\tilde{C}}{\sqrt{n}}, \quad n \geq 9. \tag{4.18}$$

Remarks. (i) *Comparison with previous work.* Research has focused on central limit theorems for $F^{(q)}_{NG_k}(\eta_s), s \to \infty$, and $F^{(q)}_{NG_k}(\xi_n), n \to \infty$, when $\mathbb{X}$ is a full-dimensional subset of $\mathbb{R}^d$ and where d is the usual Euclidean distance. This includes the seminal work [BB83], the paper [AB93] and the more recent works [PR08; PY01; PY05]. When $\mathbb{X}$ is a sub-manifold of $\mathbb{R}^d$ equipped with the Euclidean metric on $\mathbb{R}^d$, the paper [PY13] develops the limit theory for $F^{(q)}_{NG_k}(\eta_s), s \to \infty$, and $F^{(q)}_{NG_k}(\xi_n), n \to \infty$. When $\mathbb{X}$ is a compact convex subset of $\mathbb{R}^d$, the paper [LPS16] establishes the presumably optimal $O(s^{-1/2})$ rate of normal convergence for $F^{(q)}_{NG_k}(\eta_s)$.

The rate for binomial input (4.18) improves upon the rate of convergence in the Wasserstein distance d_W given by

$$d_W\left(\frac{F^{(q)}_{NG_k}(\xi_n)-\mathbf{E}F^{(q)}_{NG_k}(\xi_n)}{\sqrt{\mathrm{Var}F^{(q)}_{NG_k}(\xi_n)}},N\right)=O\left(\frac{k^4\tilde{\gamma}_p^{2/p}}{n^{(p-8)/2p}}+\frac{k^3\tilde{\gamma}_p^{3/p}}{n^{(p-6)/2p}}\right),\tag{4.19}$$

as in Theorem 3.4 of [Cha08] as well as the same rate in the Kolmogorov distance as in Section 6.3 of [LP17]. Here $\tilde{\gamma}_p := \mathbf{E}|n^{q/\gamma}S^{(q)}(X_1,\xi_n)|^p$ and $p>8$. For all $\varepsilon>0$ we have $\mathbf{P}(n^{q/\gamma}S^{(q)}(X_1,\xi_n)>\varepsilon)=(1-C\varepsilon^\gamma/n)^n$ and it follows that $\tilde{\gamma}_p^{1/p}\uparrow\infty$ as $p\to\infty$. Thus by letting $p\to\infty$, we do not recover the $O(n^{-1/2})$ rate in (4.19), but only achieve the rate $O(n^{-1/2}(\log n)^\tau)$ with some $\tau>0$.

However, the discussed papers neither provide the presumably optimal $O(n^{-1/2})$ rate of normal convergence for $F^{(q)}_{NG_k}(\xi_n)$ in the d_K distance, nor do they consider input on arbitrary metric spaces. Theorem 23 rectifies this.

(ii) *Variance bounds.* When $\mathbb{X}$ is a full-dimensional compact convex subset of $\mathbb{R}^d$, then $\gamma=d$, $\mathrm{Var}(F^{(q)}_{NG_k}(\eta_s))=\Theta(s^{1-2q/\gamma})$, and $\mathrm{Var}(F_{NG_k}(\xi_n))=\Theta(n^{1-2q/\gamma})$, which follows from Theorem 2.1 and Lemma 6.3 of [PY01] (these results treat the case $q=1$ but the proofs easily extend to arbitrary $q\in(0,\infty)$). Thus we obtain the required variance lower bounds of Theorem 23. If $\mathrm{Var}(F^{(q)}_{NG_k}(\eta_s))=\Omega(s^{1-2q/\gamma})$ does not hold, then the convergence rate in (4.17) is replaced by (3.13) with $I_{K,s}$ set to s, with a similar statement if $\mathrm{Var}(F_{NG_k}(\xi_n))=\Omega(n^{1-2q/\gamma})$ does not hold.

(iii) *Extension of Theorem 23.* The *directed* k-nearest neighbors graph, denoted $NG'_k(\zeta)$, is the directed graph with vertex set $\mathbb{X}$ obtained by including a directed edge from each point to each of its k nearest neighbors. The total edge length of the directed k-nearest neighbors graph on ζ with qth power-weighted edges is

$$F^{(q)}_{NG'_k}(\zeta)=\sum_{x\in\zeta}\tilde{S}^{(q)}(x,\zeta)$$

where

$$\tilde{S}^{(q)}(x,\zeta):=\sum_{y\in V_k(x,\zeta)}\mathsf{d}(x,y)^q.$$

The proof of Theorem 23 given below shows that the analogs of (4.17) and (4.18) hold for $F^{(q)}_{NG'_k}(\eta_s)$ and $F^{(q)}_{NG'_k}(\xi_n)$ as well.

b. *Statistics of high-dimensional data sets.* In the case that $\mathbb{X}$ is an m-dimensional $\mathscr{C}^1$-submanifold of $\mathbb{R}^d$, with d the Euclidean distance in $\mathbb{R}^d$, the *directed* nearest neighbors graph version of Theorem 23 (cf. Remark (iii) above) may be refined to give rates of normal convergence for statistics of high-dimensional non-linear data sets. This goes as follows. Recall that high-dimensional non-linear data sets are typically modeled as the realization of $\xi_n:=\{X_1,...,X_n\}$, with $X_i, 1\le i\le n$, i.i.d. copies of a random variable X having support on an unknown (non-linear) manifold $\mathbb{X}$ embedded in $\mathbb{R}^d$. Typically the coordinate representation of X_i is unknown, but the interpoint distances are known. Given this information, the goal is to establish estimators of global characteristics of $\mathbb{X}$, including intrinsic dimension, as well as global properties of the distribution of X, such as Rényi entropy. Recall that if the distribution of the random variable X has a Radon-Nikodym derivative f_X with respect to the uniform measure on $\mathbb{X}$, then given $\rho\in(0,\infty),\rho\neq1$, the Rényi ρ-entropy of X is

$$F_\rho(f_X):=(1-\rho)^{-1}\log\int_{\mathbb{X}}f_X(x)^\rho\,dx.$$

Let $\mathbb{X}$ be an m-dimensional subset of $\mathbb{R}^d$, $m \leq d$, equipped with the Euclidean metric d on $\mathbb{R}^d$. Henceforth, assume $\mathbb{X}$ is an *m-dimensional $\mathscr{C}^1$-submanifold-with-boundary* (see Section 2.1 of [PY13] for details and precise definitions). Let μ be a measure on $\mathbb{X}$ with a bounded density f_X with respect to the uniform surface measure on $\mathbb{X}$ such that condition (3.3) is satisfied with $\gamma := m$. Note that Example 2 (Section 2) provides conditions which guarantee that (3.3) holds. Assume f_X is bounded away from zero and infinity, and

$$\inf_x \mu(B(x,r)) \geq cr^m, \quad r \in [0, \operatorname{diam}(\mathbb{X})],$$

with some constant $c \in (0,\infty)$. The latter condition is called the 'locally conic' condition in [PY13] (cf. (2.3) in [PY13]).

Under the above conditions and given Poisson input η_s with intensity sf_X, the main results of [PY13] establish rates of normal convergence for estimators of intrinsic dimension, estimators of Rényi entropy, and for Vietoris-Rips clique counts (see Section 2 of [PY13] for precise statements). However these rates contain extraneous logarithmic factors and [PY13] also stops short of establishing rates of normal convergence when Poisson input is replaced by binomial input. In what follows we rectify this for estimators of Rényi entropy. The methods potentially apply to yield rates of normal convergence for estimators of Shannon entropy and intrinsic dimension, but this lies beyond the scope of this paper.

When f_X satisfies the assumptions stated above and is also continuous on $\mathbb{X}$, then $n^{q/m-1}F^{(q)}_{NG'_1}(\xi_n)$ is a consistent estimator of a multiple of $\int_{\mathbb{X}} f_X(x)^{1-q/m}\, dx$, as shown in Theorem 2.2 of [PY13]. The following result establishes a rate of normal convergence for $F^{(q)}_{NG'_k}(\xi_n)$ and, in particular, for the estimator $n^{q/m-1}F^{(q)}_{NG'_1}(\xi_n)$.

Theorem 24. *If $k \in \mathbf{N}$ and $q \in (0,\infty)$, then there is a constant $c \in (0,\infty)$ such that*

$$\mathsf{d}_K\left(\frac{F^{(q)}_{NG'_k}(\xi_n) - \mathbf{E}F^{(q)}_{NG'_k}(\xi_n)}{\sqrt{\operatorname{Var}F^{(q)}_{NG'_k}(\xi_n)}}, N\right) \leq \frac{c}{\sqrt{n}}, \quad n \geq 9. \tag{4.20}$$

A similar result holds if the binomial input ξ_n is replaced by Poisson input.

Remarks. (i) We have to exclude the case $q = 0$ since $F^{(0)}_{NG'_1}(\xi_n) = kn$ if $n > k$. For the Poisson case a central limit theorem still holds, but becomes trivial since we have $F^{(0)}_{NG'_1}(\eta_s) = k|\eta_s|$ if $|\eta_s| \geq k+1$.
(ii) In the same vein as described in Remark (i) following Theorem 23, Theorem 3.4 of [Cha08] yields a rate of normal convergence for $F^{(q)}_{NG'_1}(\xi_n)$ in the Wasserstein distance d_W given by the right-hand side of (4.19). However, the bound (4.20) is superior and is moreover expressed in the Kolmogorov distance d_K. When the input ξ_n is replaced by Poisson input η_s, we obtain the rate of normal convergence $O(s^{-1/2})$, improving upon the rates of [PY05; PY13].

4.3 Maximal points

Consider the cone $\mathrm{Co} = (\mathbb{R}^+)^d$ with apex at the origin of $\mathbb{R}^d$, $d \geq 2$. Given $\zeta \in \mathbf{N}$, $x \in \zeta$ is called maximal if $(\mathrm{Co} \oplus x) \cap \zeta = \{x\}$. In other words, a point $x = (x_1, ..., x_d) \in \zeta$ is maximal if there is no other point $(z_1, ..., z_d) \in \zeta$ with $z_i \geq x_i$ for all $1 \leq i \leq d$. The maximal layer $m_{\mathrm{Co}}(\zeta)$ is the collection of maximal points in ζ. Let $F_{\mathrm{Co}}(\zeta) := \operatorname{card}(m_{\mathrm{Co}}(\zeta))$. Maximal points are of broad interest in computational geometry and economics; see [CHH03; PS85; Sho83].

Put

$$\mathbb{X} := \{x \in [0,\infty)^d : \ F(x) \le 1\}$$

where $F : [0,\infty)^d \to \mathbb{R}^+$ is a strictly increasing function of each coordinate variable, satisfies $F(0) < 1$, is continuously differentiable, and has continuous partials F_i, $1 \le i \le d$, bounded away from zero and infinity. Let μ be a measure on $\mathbb{X}$ with Radon-Nikodym derivative g with respect to Lebesgue measure on $\mathbb{X}$, with g bounded away from zero and infinity. As usual, η_s is the Poisson point process with intensity $s\mu$ and ξ_n is a binomial point process of n i.i.d. points distributed according to μ.

Theorem 25. *There is a constant $c \in (0,\infty)$ such that*

$$\mathsf{d}_K\left(\frac{F_{\mathrm{Co}}(\eta_s) - \mathbf{E}F_{\mathrm{Co}}(\eta_s)}{\sqrt{\mathrm{Var}F_{\mathrm{Co}}(\eta_s)}}, N\right) \le cs^{-\frac{1}{2}+\frac{1}{2d}}, \quad s \ge 1. \tag{4.21}$$

Assuming $\mathrm{Var}F_{\mathrm{Co}}(\xi_n) = \Omega(n^{(d-1)/d})$*, the binomial counterpart to* (4.21) *holds, with η_s replaced by ξ_n.*

Remarks. (i) *Existing results.* The rates of normal convergence given by Theorem 25 improve upon those given in [BX01] for Poisson and binomial input for the bounded Wasserstein distance and in [BX06] and [Yuk15] for Poisson input for the Kolmogorov distance. While these findings are also proved via the Stein method, the local dependency methods employed there all incorporate extraneous logarithmic factors. Likewise, when $d = 2$, the paper [BHT03] provides rates of normal convergence in the Kolmogorov distance for binomial input, but aside from the special case that F is linear, the rates incorporate extraneous logarithmic factors. The precise approximation bounds of Theorem 25 remove the logarithmic factors in [BHT03; BX01; BX06; Yuk15].

(ii) We have taken $\mathrm{Co} = (\mathbb{R}^+)^{\mathrm{d}}$ to simplify the presentation, but the results extend to general cones which are subsets of $(\mathbb{R}^+)^d$ and which have apex at the origin.

4.4 Set approximation via Voronoï tessellations

Throughout this subsection let $\mathbb{X} := [-1/2, 1/2]^d, d \ge 2$, and let $A \subset \mathrm{int}(\mathbb{X})$ be a full-dimensional subset of $\mathbb{R}^d$. Let μ be the uniform measure on $\mathbb{X}$. For $\zeta \in \mathbf{N}$ and $x \in \zeta$ the Voronoï cell $C(x,\zeta)$ is the set of all $z \in \mathbb{X}$ such that the distance between z and x is at most equal to the distance between z and any other point of ζ. The collection of all $C(x,\zeta)$ with $x \in \zeta$ is called the Voronoï tessellation of ζ. The Voronoï approximation of A with respect to ζ is the union of all Voronoï cells $C(x,\zeta), x \in \zeta$, with $x \in A$, i.e.,

$$A(\zeta) := \bigcup_{x \in \zeta \cap A} C(x,\zeta).$$

In the following we let ζ be either a Poisson point process η_s, $s \ge 1$, with intensity measure $s\mu$ or a binomial point process ξ_n of $n \in \mathbf{N}$ points distributed according to μ. We are now interested in the behavior of the random approximations

$$A_s := A(\eta_s), \quad s \ge 1, \quad \text{and} \quad A'_n := A(\xi_n), \quad n \in \mathbf{N},$$

of A. Note that A_s is also called the Poisson-Voronoï approximation.

Typically A is an unknown set having unknown geometric characteristics such as volume and surface area. Notice that A_s and A'_n are random polyhedral approximations of A, with volumes closely approximating that of A as s and n become large. There is a large literature devoted to quantifying this approximation and we refer to [LP17],[Yuk15] for further discussion and references. One might also expect that $\mathcal{H}^{d-1}(\partial A_s)$ closely approximates a scalar multiple of $\mathcal{H}^{d-1}(\partial A)$, provided the latter quantity exists and is finite. This has been shown in [Yuk15]. Using Theorem 12(b) we deduce rates of normal convergence for the volume and surface area statistics of A_s and A'_n as well as $\mathrm{Vol}(A_s \Delta A)$ and $\mathrm{Vol}(A'_n \Delta A)$. Here and elsewhere in this section we abbreviate Vol_d by Vol. The symmetric difference $U \Delta V$ of two sets $U, V \subset \mathbb{R}^d$ is given by $U \Delta V := (U \setminus V) \cup (V \setminus U)$.

Theorem 26. *(a) Let $A \subset (-1/2, 1/2)^d$ be closed and such that ∂A satisfies $\overline{\mathcal{M}}^{d-1}(\partial A) < \infty$ and contains a $(d-1)$-dimensional C^2-submanifold and let $F \in \{\mathrm{Vol}, \mathrm{Vol}(\cdot \Delta A), \mathcal{H}^{d-1}(\partial \cdot)\}$. Then there is a constant $\tilde{C} \in (0, \infty)$ such that*

$$\mathsf{d}_K\left(\frac{F(A_s) - \mathbf{E}F(A_s)}{\sqrt{\mathrm{Var}F(A_s)}}, N\right) \leq \tilde{C} s^{-\frac{(d-1)}{2d}}, \quad s \geq 1, \tag{4.22}$$

and

$$\mathsf{d}_K\left(\frac{F(A'_n) - \mathbf{E}F(A'_n)}{\sqrt{\mathrm{Var}F(A'_n)}}, N\right) \leq \tilde{C} n^{-\frac{(d-1)}{2d}}, \quad n \geq 9, \tag{4.23}$$

as well as

$$\mathsf{d}_K\left(\frac{\mathrm{Vol}(A_s) - \mathrm{Vol}(A)}{\sqrt{\mathrm{Var}\mathrm{Vol}(A_s)}}, N\right) \leq \tilde{C} s^{-\frac{(d-1)}{2d}}, \quad s \geq 1, \tag{4.24}$$

and

$$\mathsf{d}_K\left(\frac{\mathrm{Vol}(A'_n) - \mathrm{Vol}(A)}{\sqrt{\mathrm{Var}\mathrm{Vol}(A'_n)}}, N\right) \leq \tilde{C} n^{-\frac{(d-1)}{2d}}, \quad n \geq 9. \tag{4.25}$$

(b) If $F = \mathrm{Vol}$ and $A \subset (-1/2, 1/2)^d$ is compact and convex, then all of the above inequalities are in force.

Remarks. (i) The bound (4.22) provides a rate of convergence for the main result of [Sch16] (see Theorem 1.1 there), which establishes asymptotic normality for $\mathrm{Vol}(A_s)$, A convex. The bound (4.22) also improves upon Corollary 2.1 of [Yuk15] which shows

$$\mathsf{d}_K\left(\frac{\mathrm{Vol}(A_s) - \mathbf{E}\mathrm{Vol}(A_s)}{\sqrt{\mathrm{Var}\mathrm{Vol}(A_s)}}, N\right) = O\left((\log s)^{3d+1} s^{-\frac{(d-1)}{2d}}\right).$$

Recall that the normal convergence of $\mathcal{H}^{d-1}(\partial A_s)$ is given in Remark (i) after Theorem 2.4 of [Yuk15] and the bound (4.22) for $F = \mathcal{H}^{d-1}(\partial \cdot)$ provides a rate for this normal convergence.
(ii) The bound (4.25) improves upon the bound of Theorem 6.1 of [LP17], which contains extra logarithmic factors, and, thus, addresses an open problem raised in Remark 6.9 of [LP17].
(iii) We may likewise deduce identical rates of normal convergence for other geometric statistics of A_s, including the total number of k-dimensional faces of A_s, $k \in \{0, 1, ..., d-1\}$, as well as the k-dimensional Hausdorff measure of the union of the k-dimensional faces of A_s (thus when $k = d - 1$, this gives $\mathcal{H}^{d-1}(\partial A_s)$). Second order asymptotics, including the requisite variance lower bounds for these statistics, are established in [TY06]. In the case of geometric statistics of A'_n, we expect similar variance lower bounds and central limit theorems.
(iv) Lower bounds for $\mathrm{Var}F(A_s)$ and $\mathrm{Var}F(A'_n)$ are essential to showing (4.22)-(4.25). We expect the order of these bounds to be unchanged if μ has a density bounded away from zero and infinity. We thus expect Theorem 27 to remain valid in this context because all other arguments in our proof hold for such μ.

4.5 Statistics of convex hulls of random point samples

In the following let A be a compact convex subset of $\mathbb{R}^d$ with non-empty interior, C^2-boundary and positive Gaussian curvature. By μ we denote the uniform measure on A. Let η_s, $s \geq 1$, be a Poisson point process with intensity measure $s\mu$ and let ξ_n, $n \in \mathbf{N}$, be a binomial point process of n independent points distributed according to μ. From now on $\mathrm{Conv}(\zeta)$ stands for the convex hull of a set $\zeta \subset \mathbb{R}^d$. The aim of this subsection is to establish rates of normal convergence for statistics of the random polytopes $\mathrm{Conv}(\eta_s)$ and $\mathrm{Conv}(\xi_n)$. We denote the number of k-faces of a polytope P by $f_k(P)$, $k \in \{0, \ldots, d-1\}$, and its intrinsic volumes by $V_i(P)$, $i \in \{1, \ldots, d\}$.

Theorem 27. *For any $V \in \{f_0, \ldots, f_{d-1}, V_1, \ldots, V_d\}$, there is a constant $C_V \in (0, \infty)$ also depending on A such that*

$$\mathsf{d}_K\left(\frac{V(\mathrm{Conv}(\eta_s)) - \mathbf{E}V(\mathrm{Conv}(\eta_s))}{\sqrt{\mathrm{Var}V(\mathrm{Conv}(\eta_s))}}, N\right) \leq C_V s^{-\frac{d-1}{2(d+1)}}, \quad s \geq 1, \tag{4.26}$$

and

$$\mathsf{d}_K\left(\frac{V(\mathrm{Conv}(\xi_n)) - \mathbf{E}V(\mathrm{Conv}(\xi_n))}{\sqrt{\mathrm{Var}V(\mathrm{Conv}(\xi_n))}}, N\right) \leq C_V n^{-\frac{d-1}{2(d+1)}}, \quad n \geq \max\{9, d+2\}. \tag{4.27}$$

Remarks. (i) *Previous work.* The asymptotic study of the statistics $C(\mathrm{Conv}(\eta_s))$ and $V(\mathrm{Conv}(\xi_n))$, $V \in \{f_0, \ldots, f_{d-1}, V_1, \ldots, V_d\}$, has a long and rich history, starting with the seminal work [RS63]. The breakthrough paper [Rei05], which relies on dependency graph methods and Voronoï cells, establishes rates of normal convergence for Poisson input and $V \in \{f_0, \ldots, f_{d-1}, V_d\}$ of the order $s^{-\frac{d-1}{2(d+1)}}$ times some power of $\log(s)$ (see Theorems 1 and 2). Still in the setting $V \in \{f_0, \ldots, f_{d-1}, V_d\}$, but with binomial input Theorem 1.2 and Theorem 1.3 of [Vu06] provide the rates of convergence $n^{-1/(d+1)+o(1)}$ for $d \geq 3$ and $n^{-1/6+o(1)}$ for $d = 2$, which improved previous bounds in [Rei05] for the binomial case, but is still weaker than (4.27). When $V \in \{f_0, \ldots, f_{d-1}, V_1, \ldots, V_d\}$ and A is the unit ball, Theorem 7.1 of [CSY13] gives a central limit theorem for $V(\mathrm{Conv}(\eta_s))$, with convergence rates involving extra logarithmic factors. Central limit theorems for intrinsic volume functionals over binomial input were derived recently (in parallel and independently of us) in [TTW18]. There, the rate of convergence is only for the Wasserstein distance and contains the additional factor $(\log n)^{3+2/(d+1)}$ compared to (4.27).
(ii) *Extensions.* Lower bounds for $\mathrm{Var}V(\mathrm{Conv}(\eta_s))$ and $\mathrm{Var}V(\mathrm{Conv}(\xi_n))$ are essential to showing (4.26) and (4.27). We expect the order of these bounds to be unchanged if μ has a density bounded away from zero and infinity. Consequently we anticipate that Theorem 27 remains valid in this context because all other arguments in our proof below also work for such a density.

Let us indicate the main steps of the proof. Details can be found in [LrSY19]. In the following we may assume without loss of generality that $\mathbf{0}$ is in the interior of A. We introduce a specific metric adapted to the problem, where balls are more flat close to ∂A. We denote by d_{max} the metric

$$\mathsf{d}_{max}(x, y) := \max\{\|x - y\|, \sqrt{|\mathsf{d}(x, A^c) - \mathsf{d}(y, A^c)|}\}, \quad x, y \in A,$$

and define for $x \in A$ and $r > 0$,

$$B_{\mathsf{d}_{max}}(x, r) := \{y \in A : \mathsf{d}_{max}(x, y) \leq r\}.$$

The following lemma, which proof is at [LSY19][p.958], ensures that the space $(A, \mathscr{B}(A), \mu)$ and the metric d_{max} satisfy condition (3.3) for $x \in A_{-\varrho}$, with $\gamma = d + 1$.

Lemma 28. *There is a constant $\varkappa > 0$ such that for all $x \in A_{-\varrho}$ and $r > 0$*

$$\limsup_{\varepsilon \to 0} \frac{\mu(B_{\mathsf{d}_{max}}(x, r+\varepsilon)) - \mu(B_{\mathsf{d}_{max}}(x,r))}{\varepsilon} \leq \varkappa(d+1)r^d. \tag{4.28}$$

For $k \in \{0, \dots, d-1\}$ and $\zeta \in \mathbf{N}$ let $\mathcal{F}_k(\mathrm{Conv}(\zeta))$ be the set of k-dimensional faces of $\mathrm{Conv}(\zeta)$. To cast $f_k(\mathrm{Conv}(\zeta))$ in the form of (3.2), we define

$$S_k(x,\zeta) := \frac{1}{k+1} \sum_{F \in \mathcal{F}_k(\mathrm{Conv}(\zeta))} \mathbf{1}_{\{x \in F\}}, \quad x \in \zeta.$$

Note that $f_k(\mathrm{Conv}(\zeta)) = \sum_{x \in \zeta} S_k(x,\zeta)$.

To cast the intrinsic volumes $V_j(\mathrm{Conv}(\zeta))$, $j \in \{1, \dots, d-1\}$, in the form of (3.2), we need some more notation. Given the convex set A and a linear subspace E, denote by $A|E$ the orthogonal projection of A onto E. For $x \in \mathbb{R}^d \setminus \{0\}$, let $L(x)$ the line spanned by x. Given a line $N \subset \mathbb{R}^d$ through the origin, and for $1 \leq j \leq d$, let $G(N,j)$ be the set of j-dimensional linear subspaces of $\mathbb{R}^d$ containing N. Let then $\nu_j^N(\cdot)$ be the Haar probability measure on $G(N,j)$. Let $M \subset A$ be convex. For $j \in \{0, \dots, d-1\}$, $x \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and $L \in G(L(x), j)$ define

$$f^L(x) := \mathbf{1}_{\{x \in (A|L) \setminus (M|L)\}}$$

and, as in [CSY13], define the *projection avoidance function* $\theta_j^{A,M} : \mathbb{R}^d \setminus \{\mathbf{0}\} \mapsto [0,1]$ by

$$\theta_j^{A,M}(x) := \int_{G(L(x),j)} f^L(x)\, \nu_j^{L(x)}(dL).$$

The following result generalizes [CSY13, (2.7)] to non-spherical compact sets, with arguments similar to Lemma A1 from [GT18]. The proof is in the appendix of [LSY19].

Lemma 29. *Let $M \subset A$ be a convex subset of $\mathbb{R}^d$. For all $j \in \{0, \dots, d-1\}$ there is a constant $\varkappa_{d,j}$ depending on d, j such that*

$$V_j(A) - V_j(M) = \varkappa_{d,j} \int_{A \setminus M} \theta_j^{A,M}(x) \|x\|^{-(d-j)}\, dx. \tag{4.29}$$

For $\zeta \in \mathbf{N}$ and $F \in \mathcal{F}_{d-1}(\mathrm{Conv}(\zeta))$ put $\mathrm{cone}(F) := \{ry : \ y \in F, r > 0\}$. Define for $j \in \{1, ..., d-1\}$

$$S_{j,s}(x,\zeta) = \frac{s\varkappa_{d,j}}{d} \sum_{F \in \mathcal{F}_{d-1}(\mathrm{Conv}(\zeta))} \mathbf{1}_{\{x \in F\}} \int_{\mathrm{Cone}(F) \cap (A \setminus \mathrm{Conv}(\zeta))} \|x\|^{-(d-j)} \theta_j^{A, \mathrm{Conv}(\zeta)}(x)\, dx$$

for $x \in \zeta$, $s \geq 1$. Lemma 29 yields

$$s(V_j(A) - V_j(\mathrm{Conv}(\zeta))) = \sum_{x \in \zeta} S_{j,s}(x,\zeta) \tag{4.30}$$

if $\mathbf{0}$ is in the interior of $\mathrm{Conv}(\zeta)$ and if all points of ζ are in general position. For $x \in \zeta$ and $s \geq 1$ define

$$S_{d,s}(x,\zeta) := \frac{s}{d} \sum_{F \in \mathcal{F}_{d-1}(\mathrm{Conv}(\zeta))} \mathbf{1}_{\{x \in F\}} \int_{\mathrm{Cone}(F) \cap (A \setminus \mathrm{Conv}(\zeta))} dx.$$

If $\mathbf{0}$ is in the interior of $\mathrm{Conv}(\zeta)$ and all points of ζ are in general position, we have as well

$$sV_d(A \setminus \mathrm{Conv}(\zeta)) = \sum_{x \in \mathbb{X}} S_{d,s}(x,\zeta).$$

The definitions of the scores and $\|\theta_j^{A,\mathrm{Conv}(\zeta)}\|_\infty \leq 1$ show that for $\zeta \in \mathbf{N}$, $x \in \mathbb{X}$, $s \geq 1$ and $j \in \{0, ..., d-1\}$

$$S_{j,s}(x,\zeta) \leq \varkappa_{d,j} r(\mathrm{Conv}(\zeta))^{-(d-j)} S_{d,s}(x,\zeta), \tag{4.31}$$

where $r(\mathrm{Conv}(\zeta))$ is the radius of the largest ball centered at $\mathbf{0}$ and contained in $\mathrm{Conv}(\zeta)$.

Since $\mathbf{0} \in \mathrm{int}(A)$, we can choose $\rho_0 \in (0, \tilde{\varrho})$ such that $B(\mathbf{0}, 2\rho_0) \subset A$. For a score S we denote by $\tilde{S}$ the modified score

$$\tilde{S}(x,\zeta) := \mathbf{1}_{\{x \in A_{-\rho_0}\}} S(x, (\zeta \cap A_{-\rho_0}) \cup \{\mathbf{0}\})$$

for $\zeta \in \mathbf{N}$ and $x \in \zeta$. Our strategy of proof for Theorem 27 is to apply in a first step Corollary 11 in connection with Remark (vii) after Theorem 12 to these modified scores, putting $\mathbb{X} := A$ and $\tilde{\mathbb{X}} := A_{-\rho_0}$ and K set to ∂A. Thereafter we show that the result remains true without truncating and without adding the origin as an additional point.

For a score S and $\zeta \in \mathbf{N}$ we define

$$\Sigma_S(\zeta) := \sum_{x \in \zeta} S(x,\zeta).$$

Lemma 30 ([LSY19], Lemma 3.9)**.** *For any $S_s \in \{S_0, \ldots, S_{d-1}, S_{1,s}, \ldots, S_{d,s}\}$ there are constants $C_0, c_0 \in (0,\infty)$ such that*

$$\begin{aligned} &\max\{\mathbf{P}(\Sigma_{S_s}(\eta_s) \neq \Sigma_{\tilde{S}_s}(\eta_s)), \mathbf{P}(B^d(\mathbf{0},\rho_0) \not\subset \mathrm{Conv}(\mathbf{P}_s)),\\ &\qquad\qquad |\mathbf{E}\Sigma_{S_s}(\eta_s) - \mathbf{E}\Sigma_{\tilde{S}_s}(\eta_s)|, |\mathrm{Var}\Sigma_{S_s}(\eta_s) - \mathrm{Var}\Sigma_{\tilde{S}_s}(\eta_s)|\}\\ &\leq C_0 \exp(-c_0 s) \end{aligned}$$

for $s \geq 1$ and

$$\begin{aligned} &\max\{\mathbf{P}(\Sigma_{S_n}(\xi_n) \neq \Sigma_{\tilde{S}_n}(\xi_n)), \mathbf{P}(B^d(\mathbf{0},\rho_0) \not\subset \mathrm{Conv}(\xi_n)),\\ &\qquad\qquad |\mathbf{E}\Sigma_{S_n}(\xi_n) - \mathbf{E}\Sigma_{\tilde{S}_n}(\xi_n)|, |\mathrm{Var}\Sigma_{S_n}(\xi_n) - \mathrm{Var}\Sigma_{\tilde{S}_n}(\xi_n)|\}\\ &\leq C_0 \exp(-c_0 n) \end{aligned}$$

for $n \geq 1$.

The results of [Rei05] show that for $S_s \in \{S_0, \ldots, S_{d-1}, S_{d,s}\}$ one has

$$\mathrm{Var}\Sigma_{S_s}(\eta_s) = \Theta(s^{\frac{d-1}{d+1}}) \quad \text{and} \quad \mathrm{Var}\Sigma_{S_n}(\xi_n) = \Theta(n^{\frac{d-1}{d+1}}). \tag{4.32}$$

For $S_s \in \{S_{1,s}, \ldots, S_{d-1,s}\}$ and taking into account scaling (4.30), we know from Corollary 7.1 of [CSY13] and from Theorem 2 of [BFV10] that

$$\mathrm{Var}\Sigma_{S_s}(\eta_s) = \Theta(s^{\frac{d-1}{d+1}}) \quad \text{and} \quad \mathrm{Var}\Sigma_{S_n}(\xi_n) = \Theta(n^{\frac{d-1}{d+1}}). \tag{4.33}$$

Hence, Lemma 30 implies that for $S_s \in \{S_0, \ldots, S_{d-1}, S_{1,s}, \ldots, S_{d,s}\}$

$$\mathrm{Var}\Sigma_{\tilde{S}_s}(\eta_s) = \Theta(s^{\frac{d-1}{d+1}}) \quad \text{and} \quad \mathrm{Var}\Sigma_{\tilde{S}_n}(\xi_n) = \Theta(n^{\frac{d-1}{d+1}}). \tag{4.34}$$

We define the map $R : A \times \mathbf{N} \to \mathbb{R}$ which sends $(x, \mathbb{X})$ to

$$\begin{aligned} &R(x, \mathcal{X} \cup \{x\}) :=\\ &\begin{cases} c_{max} \inf\{r \geq 0 : \mathbb{X} \cap A_{-\rho_0} \cap A_{x,r} \cap F_{x,i} \neq \emptyset \text{ for } i \in \{1, \ldots, 2^{d-1}\}\}, & x \in A_{-\rho_0}\\ 0, & x \notin A_{-\rho_0}. \end{cases} \end{aligned}$$

The next lemma shows that all $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ satisfy (3.6) and (3.7) with $\alpha_{stab} = d + 1$.

Lemma 31 ([LSY19], Lemma 3.11)**.** *R is a radius of stabilization for any $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ and there are constants $C, c \in (0,\infty)$ such that for $r \geq 0$, $x \in A$*

$$\mathbf{P}(R(x, \eta_s \cup \{x\}) \geq r) \leq C \exp(-csr^{d+1}), \quad s \geq 1,$$

whereas

$$\mathbf{P}(R(x, S_{n-8} \cup \{x\}) \geq r) \leq C \exp(-cnr^{d+1}), \quad n \geq 9.$$

The next lemma shows that all $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ satisfy (3.10) and (3.11) with $\alpha_{\partial A} = d+1$.

Lemma 32. *For any $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ there are constants $C_b, c_b \in (0,\infty)$ such that for $x \in A$, $\mathcal{A} \subset A$ with $|\mathcal{A}| \leq 7$*

$$\mathbf{P}(\tilde{S}_s(x, \eta_s \cup \{x\} \cup \mathcal{A}) \neq 0) \leq C_b \exp(-c_b s \mathsf{d}_{max}(x, A^c)^{d+1}), \quad s \geq 1,$$

whereas

$$\mathbf{P}(\tilde{S}_n(x, S_{n-8} \cup \{x\} \cup A) \neq 0) \leq C_b \exp(-c_b n \mathsf{d}_{max}(x, A^c)^{d+1}), \quad n \geq 9.$$

Proof. For $x \in A_{-\rho_0}$, $\mathbb{X} \in \mathbf{N}$ and $\mathcal{A} \subset A$ with $|\mathcal{A}| \leq 7$ we have that $\tilde{S}_s(x, \mathbb{X} \cup \{x\} \cup \mathcal{A}) = 0$ if $R(x, \mathbb{X} \cup \{x\}) \leq \sqrt{\mathsf{d}(x, A^c)} = \mathsf{d}_{max}(x, A^c)$. Thus, the assertions follow from Lemma 31. □

Lemma 33. *For any $q \geq 1$ and $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ there is a constant $C_q \in (0,\infty)$ such that for all $\mathcal{A} \subset A$ with $|\mathcal{A}| \leq 7$,*

$$\sup_{s \geq 1} \sup_{x \in A} \mathbf{E}|\tilde{S}_s(x, \eta_s \cup \{x\} \cup \mathcal{A})|^q \leq C_q$$

and

$$\sup_{n \in \mathbf{N}, n \geq 9} \sup_{x \in A} \mathbf{E}|\tilde{S}_n(x, S_{n-8} \cup \{x\} \cup \mathcal{A})|^q \leq C_q.$$

We finally obtain:

Lemma 34. *For any $\tilde{S}_s \in \{\tilde{S}_0, \ldots, \tilde{S}_{d-1}, \tilde{S}_{1,s}, \ldots, \tilde{S}_{d,s}\}$ there is a constant $\tilde{C} \in (0,\infty)$ such that*

$$\mathsf{d}_K\left(\frac{\Sigma_{\tilde{S}_s}(\eta_s) - \mathbf{E}\Sigma_{\tilde{S}_s}(\eta_s)}{\sqrt{\mathrm{Var}\Sigma_{\tilde{S}_s}(\eta_s)}}, N\right) \leq \tilde{C} s^{-\frac{d-1}{2(d+1)}}, \quad s \geq 1,$$

and

$$\mathsf{d}_K\left(\frac{\Sigma_{\tilde{S}_n}(\xi_n) - \mathbf{E}\Sigma_{\tilde{S}_n}(\xi_n)}{\sqrt{\mathrm{Var}\Sigma_{\tilde{S}_n}(\xi_n)}}, N\right) \leq \tilde{C} n^{-\frac{d-1}{2(d+1)}}, \quad n \geq 9.$$

Proof. By Lemmas 28, 31, 32, and 33 all conditions of Corollary 11 in connection with Remark (vii) after Theorem 12 are satisfied with $\mathbb{X} := A$, $\tilde{\mathbb{X}} := A_{-\rho_0}$ and $K := \partial A$. Note that $I_{\partial A, s} = O(s^{(d-1)/(d+1)})$, which completes the proof. □

Proof of Theorem 27. For any pair $(X, \tilde{X})$ of square integrable random variables satisfying $\mathrm{Var}X, \mathrm{Var}\tilde{X} > 0$, a straightforward computation shows that

$$\begin{aligned} &\mathsf{d}_K\left(\frac{X - \mathbf{E}X}{\sqrt{\mathrm{Var}X}}, N\right) \\ &\leq \mathsf{d}_K\left(\frac{\tilde{X} - \mathbf{E}X}{\sqrt{\mathrm{Var}X}}, N\right) + \mathbf{P}(X \neq \tilde{X}) \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{d}_K\Big(\frac{\tilde{X}-\mathbf{E}\tilde{X}}{\sqrt{\mathrm{Var}\tilde{X}}}, N\Big(\frac{\mathbf{E}X-\mathbf{E}\tilde{X}}{\sqrt{\mathrm{Var}\tilde{X}}}, \frac{\mathrm{Var}X}{\mathrm{Var}\tilde{X}}\Big)\Big) + \mathbf{P}(X \neq \tilde{X}) \\
&\leq \mathsf{d}_K\Big(\frac{\tilde{X}-\mathbf{E}\tilde{X}}{\sqrt{\mathrm{Var}\tilde{X}}}, N\Big) + \mathsf{d}_K\Big(N, N\Big(\frac{\mathbf{E}X-\mathbf{E}\tilde{X}}{\sqrt{\mathrm{Var}\tilde{X}}}, \frac{\mathrm{Var}X}{\mathrm{Var}\tilde{X}}\Big)\Big) + \mathbf{P}(X \neq \tilde{X}) \\
&\leq \mathsf{d}_K\Big(\frac{\tilde{X}-\mathbf{E}\tilde{X}}{\sqrt{\mathrm{Var}\tilde{X}}}, N\Big) + \frac{|\mathbf{E}X-\mathbf{E}\tilde{X}|}{\sqrt{\mathrm{Var}\tilde{X}}} + C\Big|\frac{\mathrm{Var}X}{\mathrm{Var}\tilde{X}} - 1\Big| + \mathbf{P}(X \neq \tilde{X}),
\end{aligned}
$$

where $N(\mu,\sigma^2)$ stands for a Gaussian random variable with mean μ and variance σ^2 and $C \in (0,\infty)$ is some universal constant. Applying this to the pairs $(X,\tilde{X}) := (\Sigma_{S_s}(\eta_s), \Sigma_{\tilde{S}_s}(\eta_s))$ and $(X,\tilde{X}) := (\Sigma_{S_n}(\xi_n), \Sigma_{\tilde{S}_n}(\xi_n))$, respectively, together with Lemma 30, Lemma 34, (4.32), (4.33), and (4.34) completes the proof. □

4.6 Clique counts in generalized random geometric graphs

Let $(\mathbb{X}_g, \mathscr{X}_g, \mu_g)$ be equipped with a semi-metric d such that (3.3) is satisfied for some γ and $\varkappa$. Moreover, let $\mathbb{M} = [0,\infty)$ be equipped with the Borel sigma algebra $\mathscr{M} := \mathscr{B}([0,\infty))$ and a probability measure ν on $(\mathbb{M},\mathscr{M})$. By μ we denote the product measure of μ_g and ν. In the following let η_s be a marked Poisson point process with intensity measure $s\mu$, $s \geq 1$, and let ξ_n be a marked binomial point process of $n \in \mathbf{N}$ points distributed according to μ.

Given $\zeta \in \mathbf{N}$, recall that $\mathbf{N}$ is the set of point configurations in $\mathbb{X}$, and a scale parameter $\beta \in (0,\infty)$, consider the graph $G(\zeta,\beta)$ on $\mathbb{X}$ with $(x_1, m_{x_1}) \in \zeta$ and $(x_2, m_{x_2}) \in \zeta$ joined with an edge iff $\mathsf{d}(x_1,x_2) \leq \beta \min(m_{x_1}, m_{x_2})$. When $m_x = 1$ for all $x \in \zeta$, we obtain the familiar geometric graph with parameter β. Alternatively, we could use the connection rule that (x_1, m_{x_1}) and (x_2, m_{x_2}) are joined with an edge iff $\mathsf{d}(x_1,x_2) \leq \beta \max(m_{x_1}, m_{x_2})$. A scale-free random graph based on this connection rule with an underlying marked Poisson point process is studied in [Hir17]. The number of cliques of order $k+1$ in $G(\zeta,\beta)$, here denoted $\mathcal{C}_k(\zeta,\beta)$, is a well-studied statistic in random geometric graphs. Recall that $k+1$ vertices of a graph form a clique of order $k+1$ if each pair of them is connected by an edge.

The clique count $\mathcal{C}_k(\zeta,\beta)$ is also a central statistic in topological data analysis. Consider the simplicial complex $\mathcal{R}^\beta(\mathbb{X})$ whose k-simplices correspond to unordered $(k+1)$-tuples of points of $\mathbb{X}$ such that any constituent pair of points (x_1, m_{x_1}) and (x_2, m_{x_2}) satisfies $\mathsf{d}(x_1,x_2) \leq \beta \min(m_{x_1}, m_{x_2})$. When $m_x = 1$ for all $x \in \mathbb{X}$ then $\mathcal{R}^\beta(\mathbb{X})$ coincides with the Vietoris-Rips complex with scale parameter β and $\mathcal{C}_k(\zeta,\beta)$ counts the number of k-simplices in $\mathcal{R}^\beta(\mathbb{X})$.

When μ is the uniform measure on a compact set $\mathbb{X}_g \subset \mathbb{R}^d$ with $\mathrm{Vol}(\mathbb{X}_g) > 0$ and $\gamma = d$, the ungainly quantity $\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})$ studied below is equivalent to the more natural clique count $\mathcal{C}_k(\eta_1 \cap s^{1/d}\mathbb{X}_g, \beta)$, where η_1 is a rate one stationary Poisson point process in $\mathbb{R}^d$ and $\eta_1 \cap s^{1/d}\mathbb{X}_g$ is its restriction to $s^{1/d}\mathbb{X}_g$.

Theorem 35. *Let $k \in \mathbf{N}$ and $\beta \in (0,\infty)$ and assume there are constants $c_1 \in (0,\infty)$ and $c_2 \in (0,\infty)$ such that*

$$
\mathbf{P}(M_x \geq r) \leq c_1 \exp(-\frac{r^{c_2}}{c_1}), \quad x \in \mathbb{X}_g,\ r \in (0,\infty). \tag{4.35}
$$

If $\inf_{s\geq 1} \mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})/s > 0$, then there is a constant $\tilde{C} \in (0,\infty)$ such that

$$
\mathsf{d}_K\left(\frac{\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma}) - \mathbf{E}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})}{\sqrt{\mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})}}, N\right) \leq \frac{\tilde{C}}{\sqrt{s}}, \quad s \geq 1. \tag{4.36}
$$

Likewise if $\inf_{n\geq 9} \mathrm{Var}\mathcal{C}_k(\xi_n, \beta n^{-1/\gamma})/n > 0$*, then there is a constant* $\tilde{C} \in (0,\infty)$ *such that*

$$\mathsf{d}_K\left(\frac{\mathcal{C}_k(\xi_n, \beta n^{-1/\gamma}) - \mathbf{E}\mathcal{C}_k(\xi_n, \beta n^{-1/\gamma})}{\sqrt{\mathrm{Var}\mathcal{C}_k(\xi_n, \beta n^{-1/\gamma})}}, N\right) \leq \frac{\tilde{C}}{\sqrt{n}}, \quad n \geq 9. \tag{4.37}$$

Remarks. (i) When $\mathbb{X}_g$ is a full-dimensional subset of $\mathbb{R}^d$ and when $M_x \equiv 1$ for all $x \in \mathbb{X}_g$, i.e., ν is the Dirac measure concentrated at one, a central limit theorem for the Poisson case is shown in [Pen03, Theorem 3.10]. Although the result in [Pen03] is non-quantitative, the method of proof should yield a rate of convergence for the Kolmogorov distance. Rates of normal convergence with respect to the Wasserstein distance d_W are given in [Dec+14].
(ii) The contributions of this theorem are three-fold. First, $\mathbb{X}$ may be an arbitrary metric space, not necessarily a subset of $\mathbb{R}^d$. Second, the graphs $G(\eta_s, \beta s^{-1/\gamma})$ and $G(\xi_n, \beta n^{-1/\gamma})$ are more general than the standard random geometric graph, as they consist of edges having arbitrary (exponentially decaying) lengths. Third, by applying our general findings we obtain presumably optimal rates of convergence for the Poisson and the binomial case at the same time.
(iii) The random variable $\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})$ is a so-called Poisson U-statistic. Bounds for the normal approximation of such random variables were deduced, for example, in [RS13] and [LP17] for the Wasserstein distance and in [Sch12] and [ET14] for the Kolmogorov distance. These results should also yield bounds similar to those in (4.36).
(iv) The assumption $\inf_{s\geq 1} \mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})/s > 0$ is satisfied if $\mathbb{X} \subset \mathbb{R}^d$ is a full d-dimensional set and g is a bounded probability density, as noted in the proof of Theorem 2.5 in Section 6 of [PY13]. If this assumption is not satisfied then we would have instead

$$\begin{aligned}&\mathsf{d}_K\left(\frac{\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma}) - \mathbf{E}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})}{\sqrt{\mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})}}, N\right)\\ &\leq \tilde{C}\left(\frac{\sqrt{s}}{\mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma})} + \frac{s}{(\mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma}))^{3/2}} + \frac{s^{3/2}}{(\mathrm{Var}\mathcal{C}_k(\eta_s, \beta s^{-1/\gamma}))^2}\right)\end{aligned}$$

for $s \geq 1$. A similar comment applies for an underlying binomial point process in the situation where $\inf_{n\geq 9} \mathrm{Var}\mathcal{C}_k(\xi_n, \beta n^{-1/\gamma})/n > 0$ does not hold.

Proof. To deduce Theorem 35 from Corollary 11, we express $\mathcal{C}_k(\zeta, \beta s^{-1/\gamma})$ as a sum of stabilizing score functions, which goes as follows. Fix $\gamma, s, \beta \in (0,\infty)$. For $\zeta \in \mathbf{N}$ and $x \in \zeta$ let $\varphi_{k,s}^{(\beta)}(x,\zeta)$ be the number of $(k+1)$-cliques containing x in $G(\zeta, \beta s^{-1/\gamma})$ and such that x is the point with the largest mark. This gives the desired identification

$$\mathcal{C}_k(\zeta, \beta s^{-1/\gamma}) = \sum_{x\in\mathbb{X}_g} \varphi_{k,s}^{(\beta)}(x,\zeta).$$

Now we are ready to deduce (4.36) and (4.37) from Corollary 11 with the scores S_s and S_n set to $\varphi_{k,s}^{(\beta)}$ and $\varphi_{k,n}^{(\beta)}$, respectively, and with K set to $\mathbb{X}_g$. Notice that $I_{K,s} = \Theta(s)$, as noted in (3.15). It is enough to show that $\varphi_{k,s}^{(\beta)}$ and $\varphi_{k,n}^{(\beta)}$ satisfy all conditions of Corollary 11. Stabilization (3.6) is satisfied with $\alpha_{stab} = a$, with the radius of stabilization

$$R_s((x,M_x), \eta_s \cup \{(x,M_x)\}) = \beta s^{-1/\gamma} M_x,$$

because M_x has exponentially decaying tails as in (4.35). For any $p > 0$ we have

$$\begin{aligned}&\mathbf{E}|\varphi_{k,s}^{(\beta)}((x,M_x), \eta_s \cup \{(x,M_x) \cup (\mathcal{A}, M_{\mathcal{A}})\})|^{4+p}\\ &\leq \mathbf{E}|\mathrm{card}\{\eta_s \cap B(x, \beta s^{-1/\gamma} M_x)\} + 7|^{(4+p)k} \leq C(\beta,p,\gamma) < \infty\end{aligned}$$

for all $x \in \mathbb{X}$, $s \geq 1$ and $\mathcal{A} \subset \mathbb{X}$ with $|\mathcal{A}| \leq 7$ and so the $(4+p)$th moment condition (3.8) holds for $p \in (0, \infty)$. The conclusion (4.36) follows from (3.16). The proof of (4.37) is similar. □

4.7 Proximity graphs

[GJL18] **Abstract:** In a proximity region graph $\mathcal{G}$ in $\mathbb{R}^d$, two distinct points x, y of a point process ζ are connected when the 'forbidden region' $S(x, y)$, determined according to a pre-established rule, has empty intersection with ζ. The Gabriel graph, where $S(x, y)$ is the open disk with diameter the line segment connecting x and y, the Voronoï tessellation graph, or the oriented nearest neighbour graph, are canonical examples. When ζ is a Poisson or binomial process, under broad conditions on the regions $S(x, y)$, bounds on the Kolmogorov and Wasserstein distances to the normal are produced for functionals of $\mathcal{G}$, including the total number of edges and the total length. Variance lower bounds, not requiring strong stabilization, are also proven to hold for a class of such functionals.

Chapter 5

Further limit theorems

5.1 Large deviations for U-statistics [LR16]

Abstract: A U-statistic of order k with kernel $f : \mathbb{X}^k \to \mathbb{R}^d$ over a Poisson process η, defined at Section 1.3.2, can also be expressed as $\sum_{(x_1,\dots,x_k)} f(x_1,\dots,x_k)$, where the summation is over k-tuples of distinct points of η, under appropriate integrability assumptions on f. U-statistics play an important role in stochastic geometry since many interesting functionals can be written as U-statistics, like intrinsic volumes of intersection processes, characteristics of random geometric graphs, volumes of random simplices, and many others. A central limit theorem and Berry-Esseen bounds can be derived from the previous section for Poisson and binomial input.

There are only few investigations concerning concentration inequalities for Poisson U-statistics. Most results concern U-statistics of order 1, i.e. linear functionals of the Poisson measure, and require a nice bound on $\sup_{\eta\in\mathbf{N}_s(\mathbb{X}),\ z\in\mathbb{X}} D_zF(\eta) < \infty$. For U-statistics of order ≥ 2 this condition is usually not satisfied, even if the kernel f is bounded. For U-statistics of order 1, this holds if $\|f\|_\infty < \infty$. In [LR16] we give concentration inequalities for U-statistics of order 1 and for higher order local U-statistics as well.

5.2 Power variation for stationary increments Lévy driven moving averages [BLP17]

Abstract: In this paper, we present some new limit theorems for power variation of k-th order increments of stationary increments Lévy driven moving averages. In the infill asymptotic setting, where the sampling frequency converges to zero while the time span remains fixed, the asymptotic theory gives novel results, which (partially) have no counterpart in the theory of discrete moving averages. More specifically, we show that the first-order limit theory and the mode of convergence strongly depend on the interplay between the given order of the increments $k \geqslant 1$, the considered power $p > 0$, the Blumenthal-Getoor index $\beta \in [0,2)$ of the driving pure jump Lévy process L and the behaviour of the kernel function g at 0 determined by the power α. First-order asymptotic theory essentially comprises three cases: stable convergence towards a certain infinitely divisible distribution, an ergodic type limit theorem and convergence in probability towards an integrated random process. We also prove a second-order limit theorem connected to the ergodic type result. When the driving Lévy process L is a symmetric β-stable process, we obtain two different limits: a central limit theorem and convergence in distribution towards a $(k-\alpha)\beta$-stable totally right skewed random variable.

5.3 Convergence fine de l'approximation Voronoï vers un ensemble fractal [LV17]

Abstract: In the paper [LV17], we study the inner and outer boundary densities of some sets with self-similar boundary having Minkowski dimension $s > d-1$ in $\mathbb{R}^d$. These quantities turn out to be crucial in some problems of set estimation, as we show here for the Voronoï approximation of the set with a random input constituted by n iid points in some larger bounded domain (The Voronoï approximation of a set is defined at Section 4.4). We prove that some classes of such sets have positive inner and outer boundary density, and therefore satisfy Berry-Esseen bounds in $n^{-s/2d}$ for Kolmogorov distance. The Von Koch flake serves as an example, and a set with Cantor boundary as a counter-example. We also give the almost sure rate of convergence of Hausdorff distance between the set and its approximation.

5.4 Mixing properties for STIT tessellations [Lac11]

The so-called STIT tessellations form the class of homogeneous (spatially stationary) tessellations of $\mathbb{R}^d$ which are stable under the nesting/iteration operation. In this paper, we establish the mixing property for these tessellations and give a precise form of the decay of

$$\frac{\mathbf{P}(A \cap \mathsf{M} = \emptyset, T_h B \cap \mathsf{M} = \emptyset)}{\mathbf{P}(A \cap \mathsf{M} = \emptyset)\mathbf{P}(B \cap \mathsf{M} = \emptyset)} - 1 \tag{5.1}$$

for A and B both compact connected sets, h a vector of $\mathbb{R}^d$, T_h the corresponding translation operator and M a stationary STIT tessellation.

5.5 Convex rearrangements of Gaussian fields [LD11]

Abstract: The monotone rearrangement of a univariate function is the non-decreasing function which yields the same image of the Lebesgue measure than the original one. The convex rearrangement of a smooth function is obtained by integrating the monotone rearrangement of its derivative. Both operators can be generalized to higher dimensions, where a monotone function is the gradient of a convex function. We define here the rearrangement of an irregular function as the limit of rearrangements of approximations, and give a consistency theorem.

In this paper, we investigate the asymptotic rearrangements for approximations of random fields. Stronger results are given for Gaussian fields, and the examples of the Levy field and the Brownian sheet are derived.

Deuxième partie

Mesures marginales pour champs aléatoires et processus ponctuels

Introduction

For $k \geqslant 1$, the k-th order marginal of a random signed measure ζ on $(\mathbb{X}, \mathscr{X})$ can be defined as

$$\nu_\zeta^k(A_1 \times \cdots \times A_k) = \mathbf{E}(\zeta(A_1)\ldots\zeta(A_k)), A_1, \in, A_k \in \mathscr{X}.$$

We leave aside the question of the right axiomatic framework for random measures, as we will exclusively consider random fields and point processes here, the reader is referred to [Kal86] for a formal presentation.

If for instance $\mathsf{f}(t), t \in \mathbb{R}^d$, is a real random field with uniformly bounded first moment, and $\zeta(A) = \int_A \mathsf{f}(t)dt, A \in \mathscr{B}_d$ bounded,

$$\nu_\zeta^k(A_1 \times \cdots \times A_k) = \mathbf{E}\left[\int_{A_1 \times \cdots \times A_k} \mathsf{f}(t_1)\ldots\mathsf{f}(t_k)dt_1\ldots dt_k\right]$$

is continuous with respect to $(\mathcal{L}^d)^k$ by Fubini's theorem. We call $\ell_\mathsf{f}^k(t_1, \ldots, t_k)$ its density, given by the Radon-Nikodym theorem for signed measures, using the σ-finiteness of $\mathcal{L}^d$.

For random measures which are typically not continuous with respect to Lebesgue measure, such as point processes, these marginal measures are usually not continuous with respect to Lebesgue measure either. Let for instance η be a random homogeneous Poisson measure on $\mathbb{R}^d$ and $k = 2$, we have for bounded measurable $A \subset \mathbb{R}^d$

$$\nu_\eta^2(\{(x,x) : x \in A\}) = \mathcal{L}^d(A).$$

For this reason we introduce the factorial moment measure $\nu^{[k]}$ for $k > 1$:

$$\nu_\eta^{[k]}(A^k) = \nu(A)(\nu(A)-1)\ldots(\nu(A)-k+1),$$

which can be equivalently characterised by

$$\nu^{[k]}(A_1 \times \cdots \times A_k) = \mathbf{E}(\eta(A_1)\ldots\eta(A_k))$$

for disjoint $A_1, \ldots, A_k \in \mathscr{B}_d$. We have for instance

$$\nu_\eta^{[k]}(A^k) = (\mathcal{L}^d(A))^k.$$

We consider in this second part two problems related to marginals of point processes, random fields, or random sets (which are viewed as $\{0,1\}$-valued random fields) on some carrier space $\mathbb{X}$, most often $\mathbb{X} \subseteq \mathbb{R}^d$. In the first part, we consider the topological characteristics of the excursion set $\mathsf{E}_u := \{x \in \mathbb{R}^d : \mathsf{f}(x) \geqslant u\}$ for $u \in \mathbb{R}$ and f a random field on $\mathbb{R}^d$. In particular, we show that under a $\mathscr{C}^{1,1}$ regularity hypothesis on f, the mean Euler characteristic of $\mathsf{E}_u \cap W$, for a polygonal bounded window $W \subset \mathbb{R}^d$, can be computed linearly from the third order marginal ℓ_f^3 in triplets of arbitrarily close points. This result is the random set counterpart of an identity between the Euler characteristic of a $\mathscr{C}^{1,1}$

manifold and its bi-covariogram. We also give in any dimension an upper bound on the number of connected components of $\mathsf{E}_u \cap W$ that allows to pass this topological identity to expectations, without requiring the triple $(\mathsf{f}(0), \partial_1 \mathsf{f}(0), \partial_2 \mathsf{f}(0))$ to have a bounded density in $\mathbb{R}^3$.

In the second chapter, we address the realisability problem of second order marginals : Given a function $\ell(x, y)$ (resp. a measure $\nu(dx, dy)$) on $\mathbb{X}^2$, is there a random set X (resp. a point process η), such that $\ell = \ell^2_{1_{\mathsf{X}}}$ (resp. $\nu^{[2]}_\eta = \nu$)? This problem has deep combinatorial implications, very hard to resolve, and one success of the method is to uncouple the realisability condition in two conditions: one being the combinatorial requirement, while the other is simply about the regularity of the candidate marginal.

Chapter 6

Euler characteristic of random sets

Introduction

Physicists and biologists are always in search of numerical indicators reflecting the microscopic and macroscopic behaviour of tissue, foams, fluids, or other spatial structures. The Euler characteristic, also called Euler-Poincaré characteristic, is a favored topological index because its additivity properties make it more manageable than connectivity indexes or Betti numbers. Given a set $A \subset \mathbb{R}^2$, let $\mathscr{G}(A)$ be the class of its bounded arc-wise connected components. We say that a set A is *admissible* if $\mathscr{G}(A)$ and $\mathscr{G}(A^c)$ are finite, and in this case its Euler characteristic is defined by

$$\chi(A) = \#\mathscr{G}(A) - \#\mathscr{G}(A^c),$$

where $\#$ denotes the cardinality of a set. Denote by $\mathscr{A}(\mathbb{R}^2)$ the class of admissible sets. It is more generally an indicator of the topological integrity of the set, as an irregular structure is more likely to be shredded in many small pieces, or pierced by many holes, which results in a large value for $|\chi(A)|$.

As an integer-valued quantity, the Euler characteristic can be easily measured and used in estimation and modelisation procedures. It is an important indicator of the porosity of a random media [Arn+05; Sch+12; Hil02], it is used in brain imagery [KF10; TW08], astronomy, [Mel90; Sch+99; Mar16], and many other disciplines. See also [Adl+10] for a general review of applied algebraic topology. In the study of parametric random media or graphs, a small value of $|\mathbf{E}\chi(A)|$ indicates the proximity of the percolation threshold, when that makes sense. See [Oku90], or [Bla+06] in the discrete setting.

The mathematical additivity property is expressed, for suitable sets A and B, by the formula

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \tag{6.1}$$

. Additivity implies that the Euler characteristic of a set can be obtained by summing infinitesimally small contributions, meaning that the Euler characteristic is intrinsically a local quantity, which global value can hence be obtained via integration agains the proper measure (called the curvature measure in $\mathbb{R}^2$). The Gauss-Bonnet theorem formalises this idea for $\mathscr{C}^2$ compact manifolds of $\mathbb{R}^2$, stating that the Euler characteristic of a smooth set is the integral along the boundary of its Gaussian curvature. Exploiting the local nature of the Euler characteristic in applications seems to be a geometric challenge, in the sense that it is not always clear how to express the mean Euler characteristic of a random set F under the form

$$\mathbf{E}\chi(F) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\mathbf{E}\varphi_\varepsilon(x,F)dx \tag{6.2}$$

where $\varphi_\varepsilon(x,F)$ only depends on $B(x,\varepsilon)\cap F$, where $B(x,\varepsilon)$ is the ball with center x and radius ε. We propose in this chapter a new formula of the form above, based on variographic tools, and valid beyond the $\mathscr{C}^2$ realm, and then apply it in a random setting.

Approach

In stochastic geometry and stereology, an important body of literature is concerned with providing formulas for computing the Euler characteristic of random sets, see for instance [HLW04; SW08; KSS06; NOP00] and references therein. Defined to be 1 for every convex body, it is extended by iterating formula (6.1)

$$\chi(\cup_i C_i) = -\sum_{I\subseteq[m],I\neq\emptyset}(-1)^{\#I}\mathbf{1}_{\{\cap_{i\in I}C_i\neq\emptyset\}}$$

for finite unions of such sets. Even though this formula seems highly non-local, it is possible to express it as a sum over local contributions using the Steiner formula, see (2.3) in [KSS06], but it is difficult to apply it under this form. There has also been an intensive research around the Euler characteristic of random fields excursions [AS15; AB13; Mar16; EL16; AW08; TW08], based upon the works of Adler, Taylor, Sammorodnitsky, Worsley, and their co-authors, see the central monograph [AT07]. We discuss in the next chapter, based on [Lac18a], the application of the present results to level sets of random fields.

In this work, we give a relation between the Euler characteristic of a bounded subset F of $\mathbb{R}^2$ and some variographic quantities related to F: given any two orthogonal unit vectors $\mathbf{u}_1,\mathbf{u}_2$, for ε sufficiently small, we have

$$\begin{aligned}\chi(F) = \varepsilon^{-2}\Big[&\mathrm{Vol}(F\cap(F-\varepsilon\mathbf{u}_1)^c\cap(F-\varepsilon\mathbf{u}_2)^c) \\ &- \mathrm{Vol}(F^c\cap(F+\varepsilon\mathbf{u}_1)\cap(F+\varepsilon\mathbf{u}_2))\Big],\end{aligned} \tag{6.3}$$

where $\mathrm{Vol}=\mathcal{L}^2$ is the 2-dimensional Lebesgue measure. This formula is valid under the assumption that F is $\mathscr{C}^{1,1}$, i.e. that ∂F is a $\mathscr{C}^1$ submanifold of $\mathbb{R}^2$ with Lipschitz normal and finitely many connected components. See Example 43 for the application of this formula to the unit disc.

In the context of a random closed set F, call R_ε the right-hand member of (6.3). If $\mathbf{E}\left[\sup_{0\leqslant\varepsilon\leqslant 1}R_\varepsilon\right]$ is finite, the value of $\mathbf{E}\chi(\mathsf{F})$ can be obtained as $\lim_{\varepsilon\to 0}\mathbf{E}R_\varepsilon$. The main asset of this formulation regarding classical approaches is that, to compute the mean Euler characteristic, one only needs to know the third-order marginal of F, i.e. the value of

$$(x,y,z)\mapsto\mathbf{P}(x,y,z\in\mathsf{F}),$$

for x,y,z arbitrarily close. We also give similar results for the intersection $\mathsf{F}\cap W$, where F is a random regular closed set and W is a rectangular (or poly-rectangular) observation window. This step is necessary to apply the results to a stationary set sampled on a bounded portion of the plane.

In the present chapter, we apply the principles underlying these formulas to obtain the mean Euler characteristic for level sets of moving averages, also called shot noise processes, where the kernels are the indicator functions of random sets which geometry is adapted to the lattice approximation. Even though the geometry of moving averages level sets attracted interest in the recent literature [AST13; BD16b], no such result seemed to exist [1]. As a by-product, the mean Euler characteristic of the associated boolean model is also obtained.

These formulas are successfully applied to excursions of smooth random fields in the next chapter, based upon [Lac18a]. For instance, in the context of Gaussian fields excursions, one can pass (6.3) to expectations under the requirement that the underlying field is $\mathscr{C}^{1,1}$, i.e. in the context of bivariate functions $\mathscr{C}^1$ with Lipschitz derivatives, plus additional moment conditions. This improves upon the classical theory [AT07] where fields have to be of class $\mathscr{C}^2$ and satisfy a.s. Morse hypotheses. Here again, the resulting formulas only require the knowledge of the field's third order marginals for arbitrarily close arguments, even if this marginal is unbounded.

The theoretical results of Adler and Taylor [AT07] regarding the Euler characteristic of random excursions require second order differentiability of the underlying field f, but the expression of the mean Euler characteristic only involves the first-order derivatives, suggesting that second order derivatives do not matter in the computation of the Euler characteristic.

Discussion

Equality (6.3) gives in fact a direct relation between the Euler characteristic, also known as the Minkowski functional of order 0, and the function $(x, y) \mapsto \mathrm{Vol}(F \cap (F+x) \cap (F+y))$. We call the latter function *bicovariogram* of F, or variogram of order 2, in reference to the *covariogram* of F, defined by $x \mapsto \mathrm{Vol}(F \cap (F+x))$ (see [Lan02; Gal11] or [Ser82] for more on covariograms). Let σ be the normalized Haar measure on the 1-dimensional circle $\mathbb{S}^1$. The formula

$$\mathrm{Per}(F) = \lim_{\varepsilon \to 0} \int_{\mathbb{S}^1} \varepsilon^{-1} \left(\mathrm{Vol}(F) - \mathrm{Vol}(F \cap (F + \varepsilon u)) \right) \sigma(d\mathbf{u}),$$

developed in the context of random sets by Galerne [Gal11], and originating from the theory of functions of bounded variations [AFP00], gives a direct relation between the first order variogram, and the perimeter of a measurable set F, which is also the Minkowski functional of order 1 in the vocabulary of convex geometry. Completing the picture with the fact that $\mathrm{Vol}(F)$ is at the same time the second-order Minkowski functional and the variogram of order 0, it seems that covariograms and Minkowski functionals are intrinsically linked. This unveils a new field of exploration, and raises the questions of extension to higher dimensions, with higher order variograms, and all Minkowski functionals.

Another motivation of the present work is that the question of the amount of information that can be retrieved from the variogram of a set is a central topic in the field of stereology, see for instance the recent work [AB09] completing the confirmation of Matheron's conjecture. Through relation (6.3), the data of the bicovariogram function with arguments arbitrarily close to 0 is sufficient to derive its Euler characteristic, and once again the extension to higher dimensions is a natural interrogation.

Plan

The chapter is organized as follows. We give in Section 6.1 some tools of image analysis, and the framework for stating our main result, Theorem 41, which proves in particular (6.3). These results are used to derive the mean Euler characteristic of shot noise level

[1] A more general result has now been derived by Biermé and Desolneux [BD16a]

sets and boolean model with polyrectangular grain. We then provide in Theorem 46 a uniform bound for the number of connected components of a digitalized set, useful for applying Lebesgue's dominated convergence Theorem. In Section 6.2, we introduce random closed sets and the conditions under which the previous results give a convenient expression for the mean Euler characteristic. Theorem 49 states hypotheses and results for homogeneous random models.

Some notation For $x \in \mathbb{R}^2$, call $x_{[1]}, x_{[2]}$ its components in the canonical basis. Also denote, for $x, y \in \mathbb{R}^d, d \geqslant 1$, by $[x, y]$ the segment delimited by x, y, and $(x, y) = [x, y] \setminus \{x, y\}$.

6.1 Euler characteristic and image analysis

The present chapter is restricted to the dimension 2, we therefore will not go further in the algebraic topology and homology theory underlying the definition of the Euler characteristic. The aim of this section is to provide a lattice approximation A^ε of A for which $\chi(A^\varepsilon)$ has a tractable expression, and explore under what hypotheses on A we have $\chi(A^\varepsilon) \to \chi(A)$ as $\varepsilon \to 0$.

Practitioners compute the Euler characteristic of a set $F \subset \mathbb{R}^d$ from a digital lattice approximation F^ε, where ε is close to 0. The computation of $\chi(F^\varepsilon)$ is based on a linear filtering with a patch containing 2^d pixels. Determining wether $\chi(F^\varepsilon) \approx \chi(F)$ is a problem with a long history in image analysis and stochastic geometry.

For $\varepsilon > 0$, call $Z_\varepsilon = \varepsilon\mathbb{Z}^2$ the square lattice with mesh ε, and say that two points of Z_ε are neighbours if they are at distance ε (with the additional convention that a point is its own neighbour). Say that two points are connected if there is a finite path of connected points between them. If the context is ambiguous, we use the terms *grid-neighbour,grid-connected,* to not mistake it with the general $\mathbb{R}^2$ connectivity. Call $\mathscr{G}^\varepsilon(M)$ the class of finite (grid-)connected components of a set $M \subseteq Z_\varepsilon$. We define in analogy with the continuous case, for $M \subseteq Z_\varepsilon$ bounded such that $\mathscr{G}^\varepsilon(M), \mathscr{G}^\varepsilon(M^c)$ are finite,

$$\chi^\varepsilon(M) = \#\mathscr{G}^\varepsilon(M) - \#\mathscr{G}^\varepsilon(M^c),$$

where $M^c = Z_\varepsilon \setminus M$. Remark in particular that two connected components touching exclusively through a corner are not grid-connected.

Call $\mathbf{u} = \{\mathbf{u}_1, \mathbf{u}_2\}$ the two canonical unit vectors or $\mathbb{R}^2$, and define for $A \subseteq \mathbb{R}^2, x \in \mathbb{R}^2, \varepsilon > 0$,

$$\begin{aligned}
\Phi^\varepsilon_{\text{out}}(x; A) &= \mathbf{1}_{\{x \in A, x+\varepsilon\mathbf{u}_1 \notin A, x+\varepsilon\mathbf{u}_2 \notin A\}}, \\
\Phi^\varepsilon_{\text{in}}(x; A) &= \mathbf{1}_{\{x \notin A, x-\varepsilon\mathbf{u}_1 \in A, x-\varepsilon\mathbf{u}_2 \in A\}}, \\
\Phi^\varepsilon_{X}(x; A) &= \mathbf{1}_{\{x \in A; x+\varepsilon\mathbf{u}_1 \notin A, x+\varepsilon\mathbf{u}_2 \notin A, x+\varepsilon(\mathbf{u}_1+\mathbf{u}_2) \in A\}}.
\end{aligned}$$

Seeing also these functionals as discrete measures, define

$$\begin{aligned}
\Phi^\varepsilon_{\text{out}}(A) &= \sum_{x \in Z_\varepsilon} \Phi^\varepsilon_{\text{out}}(x; A), \\
\Phi^\varepsilon_{\text{in}}(A) &= \sum_{x \in Z_\varepsilon} \Phi^\varepsilon_{\text{in}}(x; A), \\
\Phi^\varepsilon_{X}(A) &= \sum_{x \in Z_\varepsilon} \Phi^\varepsilon_{X}(x; A).
\end{aligned}$$

The subscripts *in* and *out* refer to the fact that $\Phi^\varepsilon_{\text{out}}(A)$ counts the number of vertices of $A \cap Z_\varepsilon$ pointing outwards towards North-East, and $\Phi^\varepsilon_{\text{in}}(A)$ is the number of vertices pointing inwards towards South-West. Define for $A \subseteq \mathbb{R}^2$

$$\chi^\varepsilon(x; A) = \Phi^\varepsilon_{\text{out}}(x; A) - \Phi^\varepsilon_{\text{in}}(x; A).$$

The functional $\Phi^\varepsilon_X(A)$ is intended to count the number of X-*configurations*. Such configurations are a nuisance for obtaining the Euler characteristic by summing local contributions. Call $\mathscr{A}_\varepsilon$ the class of bounded $M \subseteq Z_\varepsilon$ such that $\Phi^\varepsilon_X(M) = \Phi^\varepsilon_X(M^c) = 0$, with $M^c = Z_\varepsilon \setminus M$.

Lemma 36. *For $M \in \mathscr{A}_\varepsilon$,*

$$\chi^\varepsilon(M) = \sum_{x \in Z_\varepsilon} \chi^\varepsilon(x; M). \tag{6.4}$$

Proof. It is well known that, viewing M as a subgraph of $\mathbb{Z}^2$, the Euler characteristic of M can be computed as $\chi(M) = V - E + F$ where V is the number of vertices of M, E is its number of edges, and F is the number of facets, i.e. of points $x \in M$ such that $x + \varepsilon \mathbf{u}_1, x + \varepsilon \mathbf{u}_2, x + \varepsilon(\mathbf{u}_1 + \mathbf{u}_2) \in M$. We therefore have

$$\chi^\varepsilon(M) = \sum_{x \in \mathbb{Z}^2} \left[\mathbf{1}_{\{x \in M\}} - \sum_{i=1}^2 \mathbf{1}_{\{\{x, x+\varepsilon \mathbf{u}_i\} \text{ is an edge of } M\}} + \mathbf{1}_{\{x \text{ is the bottom left corner of a facet}\}} \right].$$

For each x, the summand is in $\{-1, 0, 1\}$ and can be computed in function of the configuration $(\mathbf{1}_{\{x \in M\}}, \mathbf{1}_{\{x+\varepsilon \mathbf{u}_1 \in M\}}, \mathbf{1}_{\{x+\varepsilon \mathbf{u}_2 \in M\}}, \mathbf{1}_{\{x+\varepsilon(\mathbf{u}_1+\mathbf{u}_2) \in M\}}) \in \{0,1\}^4$. Enumerating all the possible configurations and noting that the configurations $(1,0,0,1)$ and $(0,1,1,0)$ do not occur due to the assumption $M \in \mathscr{A}_\varepsilon$, it yields that indeed only the configurations $(1,0,0,0)$ give $+1$ and only the configurations $(1,1,1,0)$ give -1, which gives the conclusion. □

This formula amounts to a linear filtering of the set by a 2×2 discrete patch, and is already known and used in image analysis and in physics on discrete images. Analogues of this formula [Arn+01; NOP00] exist also in higher dimensional grids, but the dimension 2 seems to be the only one where an anisotropic form is valid, see [Sva15] for a discussion on this topic. The anisotropy is not indispensable to the results discussed in this chapter, but gives more generality and simplifies certain formulas. An isotropic formula can be obtained by averaging over the 4 directions.

Given a subset A of $\mathbb{R}^2$ and $\varepsilon > 0$, we are interested here in the topological properties of the *Gauss digitalisation of* A, defined by $[A]^\varepsilon := A \cap Z_\varepsilon$. Define $\mathcal{P} = \varepsilon[-\frac{1}{2}, \frac{1}{2})^2$, and

$$A^\varepsilon = \bigcup_{x \in [A]^\varepsilon} (x + \mathcal{P})$$

is the *Gauss reconstruction* of A based on Z_ε. In some unambiguous cases, the notation is simplified to $[A] = [A]^\varepsilon$. In this chapter, we refer to a *pixel* as a set $\mathcal{P} + x$, for $x \in \mathbb{R}^2$ not necessarily in Z_ε.

Notation We also use the notation, for $x, y \in Z_\varepsilon$, $[\![x, y]\!] = [x, y] \cap Z_\varepsilon$, $(\!|x, y|\!) = [\![x, y]\!] \setminus \{x, y\}$.

Properties 1. *For $A \subseteq \mathbb{R}^2$, A^ε is connected in $\mathbb{R}^2$ if $[A]^\varepsilon$ is grid-connected in $\mathbb{Z}^2$. The converse might not be true because of pixels touching through a corner, but this subtlety does not play any role in this chapter, because sets with X-configurations are* *systematically discarded. We also have* $\chi^\varepsilon([A]) = \chi(A^\varepsilon)$ *if* $[A] \in \mathscr{A}_\varepsilon$ *because connected components of* A *(resp.* A^c*) can be uniquely associated to grid-connected components of* $[A]$ *(resp.* $[A^c]$*).*

Most set operations commute with the operators $(.)^\varepsilon, [\cdot]^\varepsilon$*. For any* $A, B \subseteq \mathbb{R}^2$*,* $[A \cup B]^\varepsilon = [A]^\varepsilon \cup [B]^\varepsilon, [A \cap B]^\varepsilon = [A]^\varepsilon \cap [B]^\varepsilon, [\mathbb{R}^2 \setminus A]^\varepsilon = Z_\varepsilon \setminus [A]^\varepsilon$*, and those properties are followed by the reconstructions* $(A \cup B)^\varepsilon = A^\varepsilon \cup B^\varepsilon, (A \cap B)^\varepsilon = A^\varepsilon \cap B^\varepsilon, (A \setminus B)^\varepsilon = A^\varepsilon \setminus B^\varepsilon$*.*

6.1.1 Variographic quantities

The question raised in the next section is wether, for $A \subset \mathbb{R}^2$, $\chi^\varepsilon([A]^\varepsilon) = \chi(A)$ for ε sufficiently small, and the result depends crucially on the regularity of A's boundary. A remarkable asset of formula (6.4) is its nice transcription in terms of variographic tools. Let us introduce the related notation. For $x_1, \dots, x_q, y_1, \dots, y_m \in \mathbb{R}^2$, A a measurable subset of $\mathbb{R}^2$, define the *polyvariogram* of order (q, m),

$$\delta_{x_1,\dots,x_q}^{y_1,\dots,y_m}(A) := \mathrm{Vol}((A + x_1) \cap \dots \cap (A + x_q) \cap (A + y_1)^c \cap \dots \cap (A + y_m)^c).$$

The variogram of order $(2, 0)$ is known as the covariogram of A (see [Lan02, Chap. 3.1]), and we designate here by *bicovariogram* of A the polyvariogram of order $(3, 0)$. A polyvariogram of order (q, m) can be written as a linear combination of variograms with orders $(q_i, 0)$ for appropriate numbers $q_i \leqslant q$. For instance for $x, y \in \mathbb{R}^2, A \subset \mathbb{R}^2$ measurable with finite volume, we have

$$\delta_0^{x,y}(A) = \delta_0(A) - \delta_{0,x}(A) - \delta_{0,y}(A) + \delta_{0,x,y}(A).$$

A similar notion can be defined on Z_ε endowed with the counting measure: for $M \subseteq Z_\varepsilon, x_1, \dots, x_q, y_1, \dots, y_m \in Z_\varepsilon$,

$$\tilde{\delta}_{x_1,\dots,x_q}^{y_1,\dots,y_m}(M) := \#((M + x_1) \cap \dots \cap (M + x_q) \cap (M + y_1)^c \cap \dots \cap (M + y_m)^c).$$

Lemma 36 directly yields for $M \in \mathscr{A}_\varepsilon$,

$$\chi^\varepsilon(M) = \tilde{\delta}_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(M) - \tilde{\delta}_{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}^0(M).$$

We will see in the next section that for a sufficiently regular set $F \subseteq \mathbb{R}^2$, the analogue equality $\chi(F) = \delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(F) - \delta_{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}^0(F)$ holds for ε small.

6.1.2 Euler characteristic of ρ-regular sets

It is known in image morphology that the digital approximation of the Euler characteristic is in general badly behaved when the set $F \in \mathscr{A}(\mathbb{R}^2)$ possesses some inwards or outwards sharp angles, i.e. we don't have $\chi^\varepsilon([F]^\varepsilon) \to \chi(F)$ as $\varepsilon \to 0$, the boolean model being the typical example of such a failure, see [Ser82, Chap. XIII - B.6] or [Sva14]. Sets nicely behaved with respect to digitalisation are called *morphologically open and closed* (MOC), or ρ-regular, see [Ser82, Chap.V-C],[Sva15].

Before giving the characterization of such sets, let us introduce some morphological concepts, see for instance [Lan02; Ser82] for a more detailed account of mathematical morphology. We state below results in $\mathbb{R}^d$ because the arguments are based on purely metric considerations that apply identically in any dimension.

Notation The ball with centre x and radius r in the ∞-metric $\|\cdot\|_\infty$ of $\mathbb{R}^d$ is noted $B[x, r]$. The Euclidean ball with centre x and radius r is noted $B(x, r)$. For $r > 0, A \subseteq \mathbb{R}^d$, define

$$A^{\oplus r} := \{x + y : x \in A, y \in B(0,r)\}.$$
$$A^{\ominus r} := \{x \in A : B(x,r) \subseteq A\} = ((A^c)^{\oplus r})^c.$$

We also note $\partial A, \mathrm{cl}(A), \mathrm{int}(A)$ for resp. the topological boundary, closure, and interior of a set A.

Say that a closed set F has an *inside rolling ball* if for each $x \in F$, there is a closed Euclidean ball B of radius r contained in F such that $x \in B$, and say that F has an *outside rolling ball* if $\mathrm{cl}(F^c)$ has an inside rolling ball.

A set F has *reach* at least $r \geqslant 0$ if for each point x at distance $\leqslant r$ from F, there is a unique point $y \in F$ such that $d(x,y) = d(x,F)$. We note in this case $y = \pi_F(x)$. Call *reach* of F the supremum of the $r \geqslant 0$ such that F has reach at least r. The proposition below gathers some elementary facts about sets satisfying those rolling ball properties, the proof is left to the reader.

Proposition 37. *Let $\rho > 0$ and F be a closed set of $\mathbb{R}^d$ with an inside and an outside rolling ball of radius ρ. Then there is an outwards normal vector $\mathbf{n}_F(x)$ in each $x \in \partial F$. For $r \leqslant \rho$, $B_x := B(x - r\mathbf{n}_F(x), r)$, resp. $B'_x := B(x + r\mathbf{n}_F(x), r)$ is the unique inside, resp. outside rolling ball in x. Also, $\mathrm{int}(B'_x) \subseteq F^c$. Furthermore, $\partial F, F$ and F^c have reach at least r for each $r < \rho$.*

We reproduce here partially the synthetic formulation of Blashke's theorem by Walther [Wal99], which gives a connection between rolling ball properties and the regularity of the set.

Theorem 38 (Blashke)**.** *Let F be a compact and connected subset of $\mathbb{R}^d$. Then for $\rho > 0$ the following assertions are equivalent.*

(i) ∂F is a compact $(d-1)$-dimensional $\mathscr{C}^1$ submanifold of $\mathbb{R}^2$ such that the mapping $\mathbf{n}_F(\cdot)$, which associates to $x \in \partial F$ its outward normal vector to F, $\mathbf{n}_F(x)$, is ρ^{-1}-Lipschitz,

(ii) F has inside and outside rolling ball of radius ρ,

(iii) $(F^{\ominus r})^{\oplus r} = (F^{\oplus r})^{\ominus r} = F$, $r < \rho$.

Definition 39. *Let F* be a compact set of $\mathbb{R}^d$. Assume that F has finitely many connected components and satisfies either (i),(ii) or (iii) for some $\rho_0 > 0$. Since the connected components of F are at pairwise positive distance, each of them satisfies (i),(ii), and (iii), and therefore the whole set F satisfies (i),(ii) and (iii) for some $\rho > 0$, which might be smaller than ρ_0. Such a set is said to belong to *Serra's regular class*, see the monograph of Serra [Ser82]. We will say that such a set is *ρ-regular*, or simply *regular*.

Coming back to the Euler characteristic of smooth sets of $\mathbb{R}^2$, the following assumption needs to be in order for the restriction of a ρ-regular set to a polyrectangle to be topologically well behaved.

Assumption 40. *Let F be a ρ-regular set, and $W \in \mathscr{W}$. Assume that $\partial F \cap \mathrm{corners}(W) = \emptyset$ and that for $x \in \partial F \cap \partial W$, $\mathbf{n}_F(x)$ is not collinear with $\mathbf{n}_W(x)$.*

If a ρ-regular set F and a polyrectangle W do not satisfy this assumption, $F \cap W$ might have an infinity of connected components, which makes the Euler characteristic not properly defined. Let us prove that the digitalisation is consistent if this assumption is in order.

Theorem 41. *Let F be a ρ-regular set of $\mathbb{R}^2$, $W \in \mathscr{W}$ satisfying Assumption 40 such that $F \cap W$ is bounded. Then $F \cap W \in \mathscr{A}$ and there is $\varepsilon(F, W) > 0$ such that for $\varepsilon < \varepsilon(F, W)$,*

$$\begin{aligned}
\chi(F \cap W) &= \chi^{\varepsilon}([F \cap W]^{\varepsilon}) \\
&= \sum_{x \in \varepsilon\mathbb{Z}^2} \chi^{\varepsilon}(x; F \cap W) \qquad (6.5) \\
&= \varepsilon^{-2} \int_{\mathbb{R}^2} \chi^{\varepsilon}(x; F \cap W) dx \\
&= \varepsilon^{-2} \left(\delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F \cap W) - \delta^0_{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F \cap W)\right). \qquad (6.6)
\end{aligned}$$

Also, $\varepsilon(F, W) = \varepsilon(F + x, W + x)$ for $x \in \mathbb{R}^2$.

The proof is at [Lac18b], Section 3.2.

Remark 42. (i) The apparent anisotropy of (6.5)-(6.6) can be removed by averaging over all pairs $\{\mathbf{u}_1, \mathbf{u}_2\}$ of orthogonal unit vectors of $\mathbb{R}^2$. Even though (6.6) does not involve the discrete approximation, a direct proof not exploiting lattice approximation is not available, and such a proof might shed light on the nature of the relation between covariograms and Minkowski functionals.

(ii) The fact that the Euler characteristic of a regular set digitalisation converges to the right value is already known, see [Sva15, Section 6] and references therein, but it is reproved in [Lac18b], under a slightly stronger form. One of the difficulties of the proof of Theorem 41 is to deal with the intersection points of ∂W and ∂F.

(iii) It is proved in Svane [Sva15] that in higher dimensions, Euler characteristic and Minkowski functionals of order $d - 2$ can be approximated through isotropic analogues of formula (6.4). The arguments are purely metric and should be generalizable to higher dimensions. On the other hand, dealing with boundary effects in higher dimensions might be a headache.

(iv) It is clear throughout the proof that the value $\varepsilon(F, W)$ above for which (6.5)-(6.6) is valid is a continuous function of ρ, the distances between the connected components $F \cap W$, the distances between the points of $(\partial W \cap \partial F) \cup (\text{corners}(W) \cap F)$, and the angles between $\mathbf{n}_W(x)$ and $\mathbf{n}_F(x)$ at points $x \in \partial F \cap \partial W$.

Example 43. Before giving the proof, let us give an elementary graphical illustration of (6.6) with $F = B(0, 1)$ in $\mathbb{R}^2$. Let $\varepsilon > 0$. We note $\Gamma_+ = F \cap (F + \varepsilon\mathbf{u}_1)^c \cap (F + \varepsilon\mathbf{u}_2)^c$ and $\Gamma_- = F^c \cap (F + \varepsilon\mathbf{u}_1) \cap (F + \varepsilon\mathbf{u}_2)$. We should have for ε small

$$1 = \chi(F) = \varepsilon^{-2} \left[\delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F) - \delta^0_{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F)\right],$$

$\text{Vol}(\Gamma_-) = \delta^0_{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F)$, and $\text{Vol}(\Gamma_+) = \delta_0^{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F) = \delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F)$. The notation a, b, c, d, e, f designate six distinct subsets (see Figure 6.1, below) such that $\Gamma_- = a \cup b \cup c, \Gamma_+ = d \cup e \cup f$. Symmetry arguments yield that $\text{Vol}(a) = \text{Vol}(f), \text{Vol}(a \cup b) = \text{Vol}(c), \text{Vol}(f) = \text{Vol}(d \cup e)$, whence

$$\begin{aligned}
\text{Vol}(\Gamma_+) - \text{Vol}(\Gamma_-) &= \text{Vol}(a) + \text{Vol}(b) + \text{Vol}(c) - \text{Vol}(d) - \text{Vol}(e) - \text{Vol}(f) \\
&= 2(\text{Vol}(f) + \text{Vol}(b)) - \text{Vol}(d) - \text{Vol}(e) - \text{Vol}(f) \\
&= \text{Vol}(f) - \text{Vol}(d) + 2\text{Vol}(b) - \text{Vol}(e) \\
&= 2\text{Vol}(b).
\end{aligned}$$

The shape of b is very close to that of a cube with diagonal length ε, i.e. with side length $2^{-1/2}\varepsilon$. Therefore $\text{Vol}(b) \approx \varepsilon^2/2$, which confirms $1 = \chi(F) = \varepsilon^{-2} \left[\delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F) - \delta^0_{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F)\right]$.

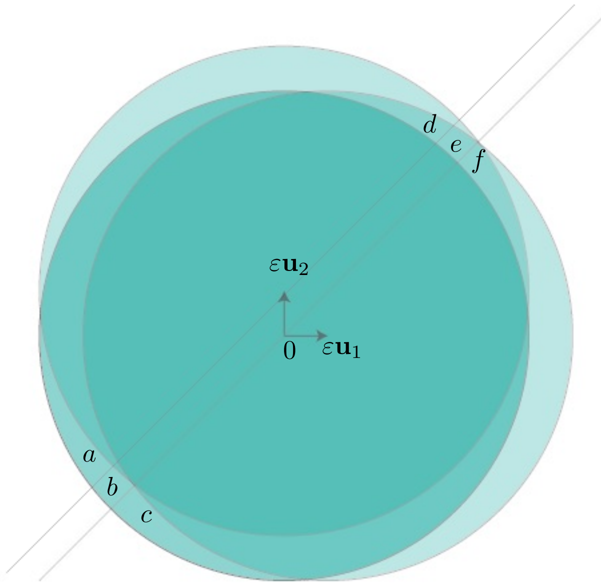


Figure 6.1 – Bicovariograms of the unit disc

Remark 44. (i) Theorem 41 still holds if ∂F intersects the outwards corner of W. In particular, we can drop the corner-related part of Assumption 40 if W is a rectangle. This subtlety makes the proof slightly more complicated, and such generality is not necessary in this chapter.

(ii) It should be possible to show that under the conditions of Theorem 41, $F \cap W$ and $(F \cap W)^\varepsilon$ are homeomorphic, but we are only interested in the Euler characteristic in this chapter.

Remark 45. It seems difficult to deal with $\mathscr{C}^1$ manifolds that don't have a Lipschitz boundary, in a general setting. Consider for instance in $\mathbb{R}^2$

$$A = \bigcup_{n=2}^{\infty} B((1/n, 0), 1/n^2).$$

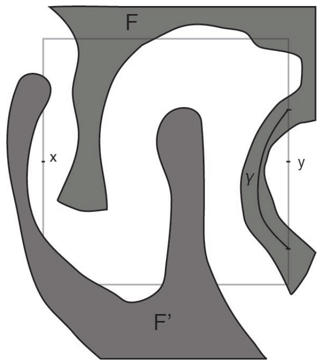
Then ∂A is a $\mathscr{C}^\infty$ embedded sub manifold of $\mathbb{R}^2$, but it has infinitely many connected components, which puts A off the class $\mathscr{A}(\mathbb{R}^2)$ of sets that we consider admissible for computing the Euler characteristic.

To have the convergence of Euler characteristic's expectation for random regular sets, we need the domination provided by Theorem 46 in the next section.

6.1.3 Bounding the number of components

Taking the expectation in formula (6.5) and switching with the limit $\varepsilon \to 0$ requires a uniform upper bound in ε on the right hand side. For ε small, (6.5) consists of a lot of positive and negative terms that cancel out. Since grouping them manually is quite intricate, this formula is not suitable for obtaining a general upper bound on $|\chi(F^\varepsilon)|$. The most efficient way consists in bounding the number of components of F^ε and $(F^c)^\varepsilon$ in terms of the regularity of the set.

Figure 6.2 – *Entanglement point* In this example, $\{x, y\} \in \mathscr{N}_\varepsilon(F)$ because the two connected components of $P'_{x,y}$, in lighter grey, are connected through $\gamma \subseteq (F \cap P_{x,y})$. We don't have $\{x, y\} \in \mathscr{N}_\varepsilon(F')$.



The result derived below is intended to be applied to ρ-regular sets, but we cannot make any assumption on the value of the regularity radius ρ, because the bound must be valid for every realization. We therefore give an upper bound on $\#\mathscr{G}(F^\varepsilon)$ and $\#\mathscr{G}((F^\varepsilon)^c)$ valid for any measurable set F.

The formula provides bounds on the number of connected components, which is a global quantity, in terms of occurrences of local configurations of the set, that we call *entanglement points*. Roughly, an entanglement occurs if two points of F^c are close but separated by a tight portion of F, see Figure 6.2. This might create disconnected components of F^ε in this region although F is locally connected.

To formalise this notion, let $x, y \in Z_\varepsilon$ grid neighbours. Introduce $P_{x,y} \subseteq \mathbb{R}^2$ the closed square with side length ε such that x and y are the midpoints of two opposite sides. Denote $P'_{x,y} = \partial P_{x,y} \setminus \{x, y\}$, which has two connected components. Then $\{x, y\}$ is an *entanglement pair of points* of F if $x, y \notin F$ and $(P'_{x,y} \cup F) \cap P_{x,y}$ is connected. We call $\mathscr{N}_\varepsilon(F)$ the family of such pairs of points.

For the boundary version, given $W \in \mathscr{W}$, we also consider grid points $x, y \in [W \cap F]$, on the same line or column of Z_ε, such that

- x, y are within distance ε from one of the edges of W (the same edge for x and y)
- $\llparenthesis x, y \rrparenthesis \neq \emptyset$
- $\llparenthesis x, y \rrparenthesis \subseteq [F^c \cap F^{\oplus\varepsilon}]$.

The family of such pairs of points $\{x, y\}$ is noted $\mathscr{N}_\varepsilon'(F; W)$.

Even though $\mathscr{N}_\varepsilon(F)$ and $\mathscr{N}_\varepsilon'(F; W)$ are not points but pairs of points of Z_ε, for $A \subseteq \mathbb{R}^2$, we extend the notation $\mathscr{N}_\varepsilon(F) \subseteq A$, (resp. $\mathscr{N}_\varepsilon(F) \cap A$), to indicate that the points of the pairs of $\mathscr{N}_\varepsilon(F)$ are contained in A (resp. the collection of pairs of points from $\mathscr{N}_\varepsilon(F)$ where both points are contained in A), and idem for $\mathscr{N}_\varepsilon'(F, W)$.

For $\{x, y\} \in \mathscr{N}_\varepsilon(F), [x, y] \cap F \neq \emptyset$ and $x, y \in F^c$. Therefore $\mathscr{N}_\varepsilon(F) \subseteq \partial F^{\oplus\varepsilon}$. We have also $\mathscr{N}_\varepsilon(F, W) \subseteq (\partial F^{\oplus\varepsilon} \cap \partial W^{\oplus\varepsilon})$.

Theorem 46. *Let F be a bounded measurable set. Then*

$$\#\mathscr{G}(F^\varepsilon) \leqslant 2\#\mathscr{N}_\varepsilon(F) + \#\mathscr{G}(F) \tag{6.7}$$

and for any $W \in \mathscr{W}$,

$$\#\mathscr{G}((F \cap W)^\varepsilon) \leqslant 2\#\mathscr{N}_\varepsilon(F) \cap W^{\oplus\varepsilon} + 2\#\mathscr{N}_\varepsilon'(F, W) + \#\mathscr{G}(F \cap W) + 2\#\text{corners}(W). \tag{6.8}$$

The proof is at [Lac18b], Section 3.3.

Remark 47. Properties 1 and (6.8) entail

$$\begin{aligned}\#\mathscr{G}(((F\cap W)^\varepsilon)^c) &= \#\mathscr{G}(((F\cap W)^c)^\varepsilon) = \#\mathscr{G}(((F^c\cap W)\cup W^c)^\varepsilon)\\ &= \#\mathscr{G}((F^c\cap W)^\varepsilon \cup (W^c)^\varepsilon) \leqslant \#\mathscr{G}((F^c\cap W)^\varepsilon) + \#\mathscr{G}((W^c)^\varepsilon)\end{aligned}$$

because adding a connected set B to a given set A can only decrease its number of bounded connected components, or increase it by 1 if B is bounded. It is easy to see that $\#\mathscr{G}((W^c)^\varepsilon) \leqslant \#\text{corners}(W)$ for ε sufficiently small. It follows that

$$\begin{aligned}|\chi((F\cap W)^\varepsilon)| \leqslant &\max(\#\mathscr{G}((F\cap W)^\varepsilon), \#\mathscr{G}((F\cap W)^\varepsilon)^c)\\ \leqslant &3\#\text{corners}(W) + 2\max(\#\mathscr{N}_\varepsilon(F)\cap W^{\oplus\varepsilon}, \#\mathscr{N}_\varepsilon(F^c)\cap W^{\oplus\varepsilon}) \\ &+ 2\max(\#\mathscr{N}_\varepsilon'(F,W), \#\mathscr{N}_\varepsilon'(F^c,W)) + \max(\#\mathscr{G}(F^c\cap W), \#\mathscr{G}(F\cap W)).\end{aligned} \tag{6.9}$$

Remark 48. The boundary of a ρ-regular set A is a $\mathscr{C}^1$ manifold, and can therefore be written under the form $\partial A = f^{-1}(\{0\})$, and $\text{cl}(A) = \{f \leqslant 0\}$ for some $\mathscr{C}^1$ function f such that $\nabla f \neq 0$ on ∂A and $\|\nabla f\|^{-1}\nabla f$ is ρ^{-1}-Lipschitz on ∂A. Such a function is said to be of class $\mathscr{C}^{1,1}$. One can bound the right hand members of (6.7)-(6.8) by quantities depending solely on f. For instance, it is proved in the next chapter that in the context of Gaussian fields, $\mathbf{E}$ and $\lim_\varepsilon$ can be switched in (6.5) if the derivatives of f are Lipschitz and their Lipschitz constants have a finite moment of sufficiently high order.

6.2 Random sets

Let $(\Omega, \mathscr{A}, \mathbf{P})$ be a complete probability space. Call $\mathscr{F}$ the class of closed sets of $\mathbb{R}^2$, endowed with the σ-algebra $\mathscr{B}(\mathscr{F})$ generated by events $\{G\cap F \neq \emptyset, F\in\mathscr{F}\}$, for G open. A $\mathscr{B}(\mathscr{F})$-measurable mapping $A : \Omega \to \mathscr{F}$ is called a Random Closed Set (RACS). See [Mol05] for a more detailed account on RACS. The functional χ is not properly defined, and therefore not measurable, on $\mathscr{F}$. We introduce the subclass $\mathscr{R}$ of regular closed sets as defined in Definition 39, and endow $\mathscr{R}$ with the trace topology and Borel σ-algebra, a random regular set being a RACS a.s. in $\mathscr{R}$. Taking the limit in $\varepsilon\to 0$ in formula (6.5) entails that χ is measurable $\mathscr{R}\to\mathbb{R}$ (the functionals $F\mapsto \#\mathscr{G}(F)$ and $F\mapsto\#\mathscr{G}(F^c)$ are also measurable). If a random regular set F satisfies a.s. Assumption 40 with some $W\in\mathscr{W}$, then $\#\mathscr{G}(F\cap W), \#\mathscr{G}((F\cap W)^c)$ and $\chi(F\cap W)$ are also measurable quantities.

Introduce the support $\text{supp}(A)$ of a RACS A as the smallest closed set K such that $\mathbf{P}(A\subseteq K) = 1$. Mostly for simplification purpose, we will assume whenever relevant that $\text{supp}(A)$ is bounded.

It is easy to derive a result giving the mean Euler characteristic as the limit of the right hand side expectation in (6.5) by combining Theorems 41 and 46. We treat below the example of stationary random sets, i.e. which law is invariant under the action of the translation group. A non-trivial stationary RACS F is a.s. unbounded, therefore we must consider the restriction of F to a bounded window W. The main issue is to handle boundary terms stemming from the intersection. They involve the perimeter of W and the specific perimeter of F. We introduce the square perimeter Per_∞ of a measurable set A with finite Lebesgue measure by the following. Note $\mathscr{C}_c^1$ the class of compactly supported functions of class $\mathscr{C}^1$ on $\mathbb{R}^2$, and define $\text{Per}_\infty(A) = \text{Per}_{\mathbf{u}_1}(A) + \text{Per}_{\mathbf{u}_2}(A)$, where

$$\text{Per}_{\mathbf{u}}(A) = \sup_{\varphi\in\mathscr{C}_c^1 : |\varphi(x)|\leqslant 1} \int_A \langle\nabla\varphi(x), \mathbf{u}\rangle dx = \text{Per}_{-\mathbf{u}}(A),\ \mathbf{u}\in\mathbb{S}^1,$$

so that we also have the expression

$$\mathrm{Per}_\infty(A) = \sup_{\varphi \in \mathscr{C}_c^1 : \|\varphi(x)\|_\infty \leqslant 1} \int_A \mathrm{div}(\varphi)(x)dx.$$

The classical variational perimeter is defined by

$$\mathrm{Per}(A) = \frac{1}{4}\int_{\mathbb{S}^1} \mathrm{Per}_{\mathbf{u}}(A)\sigma(d\mathbf{u}),$$

with the renormalized Haar measure σ on the unit circle, it satisfies $\mathrm{Per}(A) \leqslant \mathrm{Per}_\infty(A) \leqslant \sqrt{2}\mathrm{Per}(A)$. We have for instance for the square $W = [0,a]^2$, $\mathrm{Per}_\infty(W) = 4a$ and for a ball B with unit diameter in $\mathbb{R}^2$, $\mathrm{Per}_\infty(B) = 4$. See Section 9.2.2 for more on anisotropic perimeters.

It is proved in [Gal11, (1)] that for any bounded measurable set A,

$$\mathrm{Per}_{\mathbf{u}}(A) = 2\lim_{\varepsilon \to 0} \varepsilon^{-1}\delta_0^{\varepsilon \mathbf{u}}(A) = 2\lim_{\varepsilon \to 0} \varepsilon^{-1}\delta_0^{-\varepsilon \mathbf{u}}(A)$$

and in [Gal14, Proposition 16-(8)] that for any RACS F with compact support

$$\begin{aligned}\mathbf{E}\mathrm{Per}_\infty(F) &= 2\sum_{i=1}^2 \lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbf{E}\delta_0^{\varepsilon \mathbf{u}_i}(A) = 2\sum_{i=1}^2 \lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbf{E}\delta_0^{-\varepsilon \mathbf{u}_i}(A)\\ &= 2\sum_{i=1}^2 \lim_{\varepsilon \to 0} \varepsilon^{-1}\int_{\mathbb{R}^2} \mathbf{P}(x \in A, x + \varepsilon \mathbf{u}_i \notin A)dx. \end{aligned} \tag{6.10}$$

An important feature of this formula is that the mean perimeter can be deduced from the second order marginal distribution $(x, y) \mapsto \mathbf{P}(x, y \in F)$ in a neighbourhood of the diagonal $\{(x, x); x \in \mathbb{R}^2\}$. The formulas above provide a strong connection between the perimeter, called first-order Minkowski functional in the realm of convex geometry, the covariogram, and the second order marginal of a random set.

The results in the present section emphasize the connection between the Euler characteristic, Minkowski functional of order 0, and the bicovariogram, a functional that can be expressed in function of the third order marginal of a random set, in a neighbourhood of the diagonal $\{(x, x, x); x \in \mathbb{R}^2\}$.

The results below have been designed to provide an application in the context of random functions excursions, a field which has been been the subject of intense research recently, see the references in the introduction. We show in the next chapter that the quantities in (6.11) can be bounded by finite quantities under some regularity assumptions on the underlying field, and give explicit mean Euler characteristic for some stationary Gaussian fields.

Say that a closed set F is locally regular if for any compact set W, there is a ρ-regular set F' such that $F \cap W = F' \cap W$.

Proposition 49. *Let F be a stationary random closed set, a.s. locally regular, and $W \in \mathscr{W}$ bounded. Assume that the following local expectations are finite:*

$$\mathbf{E}\left[\sup_{0\leqslant\varepsilon\leqslant 1} \#\mathscr{N}_\varepsilon(F)\cap W\right], \mathbf{E}\left[\sup_{0\leqslant\varepsilon\leqslant 1} \#\mathscr{N}_\varepsilon(F^c)\cap W\right], \mathbf{E}\left[\sup_{0\leqslant\varepsilon\leqslant 1} \mathscr{N}_\varepsilon'(F, W)\right], \mathbf{E}\left[\sup_{0\leqslant\varepsilon\leqslant 1} \mathscr{N}_\varepsilon'(F^c, W)\right]. \tag{6.11}$$

Then $\mathbf{E}\left[\#\mathscr{G}(F \cap W)\right] < \infty$, $\mathbf{E}\left[\#\mathscr{G}((F \cap W)^c)\right] < \infty$, and the following limits are finite

$$\begin{aligned}\overline{\chi}(F) &:= \lim_{\varepsilon \to 0} \varepsilon^{-2}\left[\mathbf{P}(0 \in F, \varepsilon\mathbf{u}_1 \notin F, \varepsilon\mathbf{u}_2 \notin F) - \mathbf{P}(0 \notin F, -\varepsilon\mathbf{u}_1 \in F, -\varepsilon\mathbf{u}_2 \in F)\right],\\ \overline{\mathrm{Per}_{\mathbf{u}_i}}(F) &:= 2\lim_{\varepsilon \to 0}\varepsilon^{-1}\mathbf{P}(0 \in F, \varepsilon\mathbf{u}_i \notin F), i = 1, 2,\\ \overline{\mathrm{Vol}}(F) &:= \mathbf{P}(0 \in F).\end{aligned}$$

We also have, with $\overline{\mathrm{Per}_\infty}(F) = \sum_{i=1}^{2} \overline{\mathrm{Per}_{\mathbf{u}_i}}(F)$,

$$\begin{aligned} \mathbf{E}\chi(F \cap W) &= \mathrm{Vol}(W)\overline{\chi(F)} + \frac{1}{4}\Big(\mathrm{Per}_{\mathbf{u}_2}(W)\overline{\mathrm{Per}_{\mathbf{u}_1}}(F) \\ &\qquad + \mathrm{Per}_{\mathbf{u}_1}(W)\overline{\mathrm{Per}_{\mathbf{u}_2}}(F)\Big) + \chi(W)\overline{\mathrm{Vol}}(F) \end{aligned} \tag{6.12}$$

$$\mathbf{E}\mathrm{Per}_\infty(F \cap W) = \mathrm{Vol}(W)\overline{\mathrm{Per}_\infty}(F) + \mathrm{Per}(W)\overline{\mathrm{Vol}}(F) \tag{6.13}$$

$$\mathbf{E}\mathrm{Vol}(F \cap W) = \mathrm{Vol}(W)\overline{\mathrm{Vol}(F)}. \tag{6.14}$$

Chapter 7

Euler characteristic of random fields excursions

Let f be a $\mathscr{C}^1$ bivariate function with Lipschitz derivatives, and $F = \{x \in \mathbb{R}^2 : f(x) \geqslant \lambda\}$ an upper level set of f, with $\lambda \in \mathbb{R}$. We present an identity giving the Euler characteristic of F in terms of its three-points indicator functions. A bound on the number of connected components of F in terms of the values of f and its gradient, valid in higher dimensions, is also derived. In dimension 2, if f is a random field, this bound allows to pass the former identity to expectations if f's partial derivatives have Lipschitz constants with finite moments of sufficiently high order, without requiring bounded conditional densities. This approach provides an expression of the mean Euler characteristic in terms of the field's third order marginal. Sufficient conditions and explicit formulas are given for Gaussian fields, relaxing the usual $\mathscr{C}^2$ Morse hypothesis.

7.1 Introduction

The geometry of random fields excursion sets has been a subject of intense research over the last two decades. Many authors are concerned with the computation of the mean [AS15; AST13; AT03; AB13] or variance [EL16; Mar16] of the Euler characteristic, denoted by χ here.

Most of the available works on random fields use the results gathered in the celebrated monograph [AT07], or similar variants. In this case, theoretical computations of the Euler characteristic emanate from Morse theory, where the focus is on the local extrema of the underlying field instead of the set itself. For the theory to be applicable, the functions must be $\mathscr{C}^2$ and satisfy the Morse hypotheses, which conveys some restrictions on the set itself.

The expected Euler characteristic also turned out to be a widely used approximation of the distribution function of the maximum of a Morse random field, and attracted much interest in this direction, see [AS15; AB13; AW08; TW08]. Indeed, for large $r > 0$, a well-behaved field rarely exceeds r, and if it does, it is likely to have a single highest peak, which yields that the level set of f at level r, when not empty, is most often simply connected, and has Euler characteristic 1. Thereby, $\mathbf{E}\chi(\{f \geqslant r\}) \approx \mathbf{P}(\sup f \geqslant r)$, which provides an additional motivation to compute the mean Euler characteristic of random fields.

Even though [AST13] provides an asymptotic expression for some classes of infinitely divisible fields, most of the tractable formulae concern Gaussian fields. One of the ambitions of this chapter is to provide a formula that is tractable in a rather general setting, and also works in the Gaussian realm. There seems to be no particular obstacle to extend these ideas to higher dimensions in a future work.

Approach and main result

Our results exploit the findings of [Lac18b] connecting smooth sets Euler characteristic and variographic tools. For some $\lambda \in \mathbb{R}$ and a bi-variate function f, define for $x \in \mathbb{R}^2$ the event

$$\delta^\eta(x, f, \lambda) = \mathbf{1}_{\{f(x) \geqslant \lambda, f(x+\eta \mathbf{u}_1) < \lambda, f(x+\eta \mathbf{u}_2) < \lambda\}}, \eta \in \mathbb{R},$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ denotes the canonical basis of $\mathbb{R}^2$, assuming f is defined in these points.

For f a random field, let $\bar{\delta}^\eta(x, \mathsf{f}, \lambda)$ denote the event $\{\delta^\eta(x, \mathsf{f}, \lambda) = 1\}$. Let us write a corollary of our main result here, a more general statement can be found in [Lac18a]Section 7.3. Denote by Vol^2 the Lebesgue measure on $\mathbb{R}^2$. For $W \subset \mathbb{R}^2$ and a function $f : W \to \mathbb{R}^2$, introduce the mapping $\mathbb{R}^2 \to \mathbb{R}^2$,

$$f_{[W]}(x) = \begin{cases} -\infty \text{ if } x \notin W \\ f(x) \text{ otherwise,} \end{cases}$$

so that the intersections of level sets of f with W are the level sets of $f_{[W]}$.

Theorem 50. *Let $W = [0, a] \times [0, b]$ for some $a, b > 0$, f be a $\mathscr{C}^1$ real random field on $\mathbb{R}^2$ with locally Lipschitz partial derivatives $\partial_1 \mathsf{f}, \partial_2 \mathsf{f}$, $\lambda \in \mathbb{R}$, and let $\mathsf{F} = \{x \in W : \mathsf{f}(x) \geqslant \lambda\}$. Assume furthermore that the following conditions are satisfied:*

(i) For some $\varkappa > 0$, for $x \in \mathbb{R}^2$, the random vector $(\mathsf{f}(x), \partial_1 \mathsf{f}(x), \partial_2 \mathsf{f}(x))$ has a density bounded by $\varkappa$ from above on $\mathbb{R}^3$.

(ii) There is $p > 6$ such that

$$\mathbf{E}[\mathrm{Lip}(\mathsf{f}, W)^p] < \infty, \ \mathbf{E}[\mathrm{Lip}(\partial_i \mathsf{f}, W)^p] < \infty, i = 1, 2,$$

where $\mathrm{Lip}(g, W)$ denotes the Lipschitz constant of a vector-valued function g on W.

Then $\mathbf{E}[\#\mathscr{G}(\mathsf{F})] < \infty, \mathbf{E}[\#\mathscr{G}(\mathsf{F}^c)] < \infty$, and

$$\mathbf{E}[\chi(\mathsf{F})] = \lim_{\varepsilon \to 0} \sum_{x \in \varepsilon \mathbb{Z}^2} [\mathbf{P}(\bar{\delta}^\varepsilon(x, \mathsf{f}_{[W]}, \lambda)) - \mathbf{P}(\bar{\delta}^{-\varepsilon}(x, -\mathsf{f}_{[W]}, -\lambda))] \tag{7.1}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{-2} \int_{\mathbb{R}^2} \Big[\mathbf{P}(\bar{\delta}^\varepsilon(x, \mathsf{f}_{[W]}, \lambda)) - \mathbf{P}(\bar{\delta}^{-\varepsilon}(x, -\mathsf{f}_{[W]}, -\lambda))\Big] dx. \tag{7.2}$$

If f is furthermore stationary, we have

$$\mathbf{E}[\chi(\mathsf{F})] = \overline{\chi}(\mathsf{f}, \lambda)\mathrm{Vol}^2(W) + \overline{\mathrm{Per}}(\mathsf{f}, \lambda)\mathrm{Per}(W) + \overline{\mathrm{Vol}^2}(\mathsf{f}, \lambda)\chi(W)$$

where the volumic Euler characteristic, perimeter and volume $\overline{\chi}, \overline{Per}, \overline{\mathrm{Vol}^2}$ are defined in Theorem 58, they only depend on the behaviour of f around the origin.

The right hand side of (7.2) is related to the *bicovariogram* of the set F, defined by

$$\delta_0^{x,y}(\mathsf{F}) = \mathrm{Vol}^2(\mathsf{F} \cap (\mathsf{F} + x)^c \cap (\mathsf{F} + y)^c), x, y \in \mathbb{R}^2, \tag{7.3}$$

in that (7.2) can be reformulated as

$$\mathbf{E}\chi(\mathsf{F}) = \lim_{\varepsilon \to 0} \varepsilon^{-2}(\mathbf{E}[\delta_0^{-\varepsilon \mathbf{u}_1, -\varepsilon \mathbf{u}_2}(\mathsf{F})] - \mathbf{E}[\delta_0^{\varepsilon \mathbf{u}_1, \varepsilon \mathbf{u}_2}(\mathsf{F}^c)]).$$

This approach highlights the fact that under suitable conditions, the mean Euler characteristic of random level sets is linear in the field's third order marginal.

We also give in Theorem 52 a bound on the number of connected components of the excursion of f, valid in any dimension, which is finer than just bounding by the number of critical points; we could not locate an equivalent result in the literature. This topological estimate is interesting in its own and also applies uniformly to the number of components of 2D-pixel approximations of the excursions of f. We therefore use it here as a majoring bound in the application of Lebesgue's theorem to obtain (7.1)-(7.2).

It is likely that the results concerning the planar Euler characteristic could be extended to higher dimensions. See for instance [Sva14], that paves the way to an extension of the results of [Lac18b] to random fields on spaces with arbitrary dimension. Also, the uniform bounded density hypothesis is relaxed and allows for the density of the $(d+1)$-tuple $(\mathsf{f}(x), \partial_1 \mathsf{f}(x), \dots, \partial_d \mathsf{f}(x))$ to be arbitrarily large in the neighbourhood of $(\lambda, 0, \dots, 0)$. Theorem 56 features a result where f is defined on the whole space and the level sets are observed through a bounded window W, as is typically the case for level sets of non-trivial stationary fields, but the intersection with ∂W requires additional notation and care. See Theorem 58 for a result tailored to deal with excursions of stationary fields.

Theorem 60 features the case where f is a Gaussian field assuming only $\mathscr{C}^{1,1}$ regularity (classical literature about random excursions require $\mathscr{C}^2$ Morse fields in dimension $d \geqslant 2$). Under the additional hypothesis that f is stationary and isotropic, we retrieve in Theorem 62 the classical results of [AT07].

Let us explore other consequences of our results. Let $h : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^1$ test function with compact support, and F as in Theorem 7.1. Using the results of the current chapter, it is shown in the follow-up article [Lac18a] that for any deterministic $\mathscr{C}^2$ Morse function f on $\mathbb{R}^2$,

$$\int_{\mathbb{R}} \chi(\mathsf{F}) h(\lambda) d\lambda = -\sum_{i=1}^{2} \int_W \mathbf{1}_{\{\nabla \mathsf{f}(x) \in Q_i\}} [h'(\mathsf{f}(x)) \partial_i \mathsf{f}(x)^2 + h(\mathsf{f}(x)) \partial_{ii} \mathsf{f}(x)] dx + \text{boundary terms} \tag{7.4}$$

where

$$Q_1 = \{(x,y) \in \mathbb{R}^2 : y < x < 0\},\ Q_2 = \{(x,y) \in \mathbb{R}^2 : x < y < 0\},$$

yielding applications for instance to shot-noise processes. In the context of random functions, no marginal density hypothesis is required to take the expectation, at the contrary of analogous results, including those from the current chapter. Biermé & Desolneux [BD16a, Section 4.1] later gave another interpretation of (7.4), showing that if it is extended to a random isotropic stationary field which gradient does not vanish a.e. a.s., it can be rewritten as a simpler expression, after appropriate integration by parts, namely

$$\mathbf{E}\left[\int_U \chi(\{\mathsf{f} \geqslant \lambda\}; U) h(\lambda) d\lambda\right] = \mathrm{Vol}^2(U) \mathbf{E}\left[h(\mathsf{f}(0))[-\partial_{11}\mathsf{f}(0) + 4\partial_{12}\mathsf{f}(0)\partial_1\mathsf{f}(0)\partial_2 f(0) \|\nabla \mathsf{f}(0)\|^{-2}]\right],$$

where U is an appropriate open set, and $\chi(\{\mathsf{f} \geqslant \lambda; U\})$ is the total curvature of the level set $\{\mathsf{f} \geqslant \lambda\}$ within U, generalizing the Euler characteristic. They obtained this result by totally different means, via an approach involving Gauss-Bonnet theorem, without any requirement on f apart from being $\mathscr{C}^2$.

7.2 Topological approximation

Let f be a function of class $\mathscr{C}^1$ over some window $W \subset \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Define

$$F := F_\lambda(f) = \{x \in W : f(x) \geqslant \lambda\}, \qquad F_{\lambda^+}(f) = \{x \in W : f(x) > \lambda\}.$$

Remark that $F_{\lambda^+}(f) = (F_{-\lambda}(-f))^c$. If we assume that ∇f does not vanish on $f^{-1}(\{\lambda\})$, then $\partial F_\lambda(f) = \partial F_{\lambda^+}(f) = f^{-1}(\{\lambda\})$, and this set is furthermore Lebesgue-negligible, as a $(d-1)$-dimensional manifold.

According to [Fed59, p. 4.20], $\partial F_\lambda(f)$ is regular in the sense that its boundary is $\mathscr{C}^1$ with Lipschitz normal, if ∇f is locally Lipschitz and does not vanish on $\partial F_\lambda(f)$. This condition is necessary to prevent F from having locally infinitely many connected components, which would make Euler characteristic not properly defined in dimension 2, see [Lac18b, Remark 2.11]. We call $\mathscr{C}^{1,1}$ function a differentiable function whose gradient is a locally Lipschitz mapping. Those functions have been mainly used in optimization problems, and as solutions of some PDEs. They can also be characterized as the functions which are locally *semiconvex* and *semiconcave*, see [CS04].

The results of [Lac18b] also yield that the Lipschitzness of ∇f is sufficient for the digital approximation of $\chi(\{f \geqslant \lambda\})$ to be valid. It seems therefore that the $\mathscr{C}^{1,1}$ assumption is the minimal one ensuring the Euler characteristic to be computable in this fashion.

7.2.1 Observation window

An aim of the present chapter is to advocate the power of variographic tools for computing intrinsic volumes of random fields excursions. Since many applications are concerned with stationary random fields on the whole plane, we have to study the intersection of excursions with bounded windows, and assess the quality of the approximation.

To this end, call *rectangle* of $\mathbb{R}^d$ any set $W = I_1 \times \cdots \times I_d$ where the I_k are possibly infinite closed intervals of $\mathbb{R}$ with non-empty interiors, and let corners(W), which number is between 0 and 2^d, be the points having extremities of the I_i as coordinates. Then call *polyrectangle* a finite union $W = \cup_i W_i$ where each W_i is a rectangle, and for $i \neq j$, $\text{corners}(W_i) \cap \text{corners}(W_j) = \emptyset$. Call $\mathscr{W}_d$ the class of polyrectangles.

For $W \in \mathscr{W}_d$ and $x \in W$, let $I_x(W) = \{1, \dots, d\}$ if $x \in \text{int}(W)$, and otherwise let $I_x(W) \subset \{1, \dots, d\}$ be the set of indices i such that $x + \varepsilon \mathbf{u}_i \in \partial W$ and $x - \varepsilon \mathbf{u}_i \in \partial W$ for arbitrarily small $\varepsilon > 0$, where $\mathbf{u}_i$ is the i-th canonical vector of $\mathbb{R}^d$. Say then that x is a k-dimensional point of W if $|I_x(W)| = k$. Denote by $\partial_k W$ the set of k-dimensional points, and call $k-$dimensional facets the connected components of $\partial_k W$. Remark that $I_x(W)$ is constant over a given facet. Note that $\partial_d W = \text{int}(W)$ and $\partial W = \cup_{k=0}^{d-1} \partial_k W$. We extend the notation $\text{corners}(W) = \partial_0 W$. An alternative definition is that a subset $F \subset W$ is a facet of W if it is a maximal relatively open subset of an affine subspace of $\mathbb{R}^d$.

Definition 51. Let $W \in \mathscr{W}_d$, and $f : W \to \mathbb{R}$ be of class $\mathscr{C}^{1,1}$. Say that f is *regular within* W at some level $\lambda \in \mathbb{R}$ if for $0 \leqslant k \leqslant d$, $\{x \in \partial_k W : f(x) = \lambda, \partial_i f(x) = 0, i \in I_x(W)\} = \emptyset$, or equivalently if for every k-dimensional facet G of W, the k-dimensional gradient of the restriction of f to G does not vanish on $f^{-1}(\{\lambda\}) \cap G$.

For such a function f in dimension 2, it is shown in the previous chapter that the Euler characteristic of its excursion set $F = F_\lambda(f) \cap W$ can be expressed by means of its bicovariograms, defined in (7.3): for $\varepsilon > 0$ sufficiently small

$$\chi(F) = \varepsilon^{-2}[\delta_0^{-\varepsilon\mathbf{u}_1, -\varepsilon\mathbf{u}_2}(F) - \delta_0^{\varepsilon\mathbf{u}_1, \varepsilon\mathbf{u}_2}(F^c)]. \tag{7.5}$$

The proof is based on the Gauss approximation of F:

$$F^\varepsilon = \bigcup_{x\in\varepsilon\mathbb{Z}^2\cap F}\left(x+\varepsilon[-1/2,1/2)^2\right).$$

According to [Lac18b, Theorem 2.7], for ε sufficiently small,

$$\begin{aligned}\chi(F) &= \chi(F^\varepsilon)\\ &= \sum_{x\in\varepsilon\mathbb{Z}^2}(\delta^\varepsilon(x,f_{[W]},\lambda)-\delta^{-\varepsilon}(x,-f_{[W]},-\lambda))\\ &= \varepsilon^{-2}\int_{\mathbb{R}^2}(\delta^\varepsilon(x,f_{[W]},\lambda)-\delta^{-\varepsilon}(x,-f_{[W]},-\lambda))dx.\end{aligned}$$

If f is a random field, the difficulty to pass the result to expectations is to majorize the right hand side uniformly in ε by an integrable quantity, and this goes through bounding the number of connected components of F and its approximation F^ε. This is the object of the next section.

7.2.2 Topological estimates

The next result, valid in dimension $d \geqslant 1$, does not concern directly the Euler characteristic. Its purpose is to bound the number of connected components of $F_\lambda(f)\cap W$ by an expression depending on f and its partial derivatives. It turns out that a similar bound holds for the excursion approximation $(F_\lambda(f)\cap W)^\varepsilon$ in dimension 2, uniformly in ε, enabling the application of Lebesgue's theorem to the point-wise convergence (7.5).

Traditionally, see for instance [EL16, Prop. 1.3], the number of connected components of the excursion set, or its Euler characteristic, is bounded by using the number of critical points, or by the number of points on the level set where f's gradient points towards a predetermined direction. Here, we use another method based on the idea that in a small connected component, a critical point is necessarily close to the boundary, where $f-\lambda$ vanishes. It yields the expression (7.6) as a bound on the number of connected components. It also allows in Section 7.3, devoted to random fields, to relax the usual uniform density assumption on the marginals of the $(d+1)$-tuple $(f,\partial_i f, i=1,\dots,d)$, leaving the possibility that the density is unbounded around $(\lambda,0,\dots,0)$.

Denote by $\mathrm{Lip}(g;A)\in\mathbb{R}_+\cup\{\infty\}$, or just $\mathrm{Lip}(g)$, the Lipschitz constant of a mapping g going from a metric space A to another metric space. Let $W\in\mathscr{W}_d, g: W\to\mathbb{R}, \mathscr{C}^1$ with Lipschitz derivatives. Denote by $\mathcal{H}^k_d$ the k-dimensional Hausdorff measure in $\mathbb{R}^d$. Define the possibly infinite quantity, for $1\leqslant k\leqslant d$,

$$I_k(g;W):=\max(\mathrm{Lip}(g),\mathrm{Lip}(\partial_i g),1\leqslant i\leqslant d)^k\int_{\partial_k W}\frac{\mathcal{H}^k_d(dx)}{\max\left(|g(x)|,|\partial_i g(x)|,i\in I_x(W)\right)^k},$$

and $I_0(g;W):=\#\mathrm{corners}(W)$. Put $I_k(g;W)=0$ if $\mathrm{Lip}(g)=0$ and g vanishes, $1\leqslant k\leqslant d$.

Theorem 52. *Let $W\in\mathscr{W}_d$, and $f: W\to\mathbb{R}$ be a $\mathscr{C}^{1,1}$ function. Let $F=F_\lambda(f)$ or $F=F_{\lambda+}(f)$ for some $\lambda\in\mathbb{R}$. Assume that f is regular at level λ in W.*

(i) For $d\geqslant 1$,

$$\#\mathscr{G}(F\cap W)\leqslant\sum_{k=0}^d 2^k\varkappa_k^{-1}I_k(f-\lambda;W),\tag{7.6}$$

where $\varkappa_k$ is the volume of the k-dimensional unit ball.

(ii) If $d = 2$,

$$\#\mathscr{G}((F \cap W)^{\varepsilon}) \leqslant C \sum_{k=0}^{2} I_k(f - \lambda; W) \tag{7.7}$$

for some $C > 0$ *not depending on* f, λ, *or* ε.

The proof is given in [Lac18a].

Remark 53. Theorem 56 gives conditions on the marginal densities of a bivariate random field ensuring that the term on the right hand side has finite expectation.

Remark 54. Similar results hold if partial derivatives of f are only assumed to be Hölder-continuous, i.e. if there is $\delta > 0$ and $H_i > 0, i = 1, \ldots, d$, such that $\|\partial_i f(x) - \partial_i f(y)\| \leqslant H_i \|x - y\|^{\delta}$ for x, y such that $[x, y] \subset W$. Namely, we have to change constants and replace the exponent k in the *max* by an exponent $k\delta$. We do not treat such cases here because, as noted at the beginning of Section 7.2, if the partial derivatives are not Lipschitz, the upper level set is not regular enough to compute the Euler characteristic from the bicovariogram, but the proof would be similar to the $\mathscr{C}^{1,1}$ case.

Remark 55. Calling B the right hand term of (7.7) and noticing that $F_{\lambda^+}(f)^c$ is an upper level set of $-f$, an easy reasoning yields (see [Lac18b, Remark 2.13])

$$|\chi((F_\lambda(f) \cap W)^{\varepsilon})| \leqslant 2B.$$

7.3 Mean Euler characteristic of random excursions

We call here $\mathscr{C}^1$ random field over a set $\Omega \subseteq \mathbb{R}^d$ a separable random field $(\mathsf{f}(x); x \in \Omega)$, such that in each point $x \in \Omega$, the limits

$$\partial_i \mathsf{f}(x) := \lim_{s \to 0} \frac{\mathsf{f}(x + s\mathbf{u}_i) - \mathsf{f}(x)}{s}, \ i = 1, 2,$$

exist a.s., and the fields $(\partial_i \mathsf{f}(x), x \in \Omega), i = 1, \ldots, d$, are a.s. separable with continuous sample paths. See [Adl81; AT07] for a discussion on the regularity properties of random fields. Say that the random field is $\mathscr{C}^{1,1}$ if the partial derivatives are a.s. locally Lipschitz.

Many sets of conditions allowing to take the expectation in (7.5) can be derived from Theorem 52. We give below a compromise between optimality and compactness.

Theorem 56. *Let* $W \in \mathscr{W}_d$ *bounded, and let* f *be a* $\mathscr{C}^{1,1}$ *random field on* W, $\lambda \in \mathbb{R}, \mathsf{F} = \{x \in W : f(x) \geqslant \lambda\}$. *Assume that the following conditions are satisfied:*

(i) For some $\varkappa > 0, \alpha > 1$, *for* $1 \leqslant k \leqslant d, x \in \partial_k W, I \subset \mathcal{I}_k$, *the random* $(k+1)$*-tuple* $(\mathsf{f}(x) - \lambda, \partial_i \mathsf{f}(x), i \in I)$ *satisfies*

$$\mathbf{P}(|\mathsf{f}(x) - \lambda| \leqslant \varepsilon, |\partial_i \mathsf{f}(x)| \leqslant \varepsilon, i \in I) \leqslant \varkappa \varepsilon^{\alpha k}, \varepsilon > 0,$$

(ii) for some $p > d\alpha(\alpha - 1)^{-1}$,

$$\mathbf{E}[\mathrm{Lip}(\mathsf{f})^p] < \infty, \ \mathbf{E}[\mathrm{Lip}(\partial_i \mathsf{f})^p] < \infty, i = 1, \ldots, d.$$

Then $\mathbf{E}[\#\mathscr{G}(\mathsf{F})] < \infty, \mathbf{E}[\#\mathscr{G}(\mathsf{F}^c)] < \infty$ *and* f *is a.s. regular within* W *at level* λ. *In the context* $d = 2$, (7.1)-(7.2) *give the mean Euler characteristic.*

Remark 57. In the case where the $\mathrm{Lip}(\mathsf{f}), \mathrm{Lip}(\partial_i \mathsf{f}), i = 1, \ldots, d$ have a finite moment of order $> d(d+1)$, the hypotheses are satisfied if for instance $(\mathsf{f}(x) - \lambda, \partial_i \mathsf{f}(x), 1 \leqslant i \leqslant d)$ has a uniformly bounded multivariate density, in which case $\alpha = (d+1)/d$ is suitable. If $\alpha < (d+1)/d$, higher moments for the Lipschitz constants are required.

We give an explicit expression in the case where f is stationary. Boundary terms involve the perimeter of F.

Theorem 58. *Let* f *be a* $\mathscr{C}^{1,1}$ *stationary random field on* $\mathbb{R}^2$, $\lambda \in \mathbb{R}$, $W \in \mathscr{W}_2$ *bounded* $\mathsf{F} = F_\lambda(\mathsf{f})$. *Assume that* $(\mathsf{f}(0), \partial_1 \mathsf{f}(0), \partial_2 \mathsf{f}(0))$ *has a bounded density, and that there is* $p > 6$ *such that*

$$\mathbf{E}\Big[\mathrm{Lip}(\mathsf{f}; W)^p\Big] < \infty,\ \mathbf{E}\Big[\mathrm{Lip}(\partial_i \mathsf{f}; W)^p\Big] < \infty, i = 1, 2.$$

Then the following limits exist:

$$\begin{aligned}
\overline{\chi}(\mathsf{f}, \lambda) &:= \lim_{\varepsilon \to 0} \varepsilon^{-2} \Big[\mathbf{P}(\bar{\delta}^{\varepsilon}(0, \mathsf{f}, \lambda)) - \mathbf{P}(\bar{\delta}^{-\varepsilon}(0, -\mathsf{f}, -\lambda))\Big],\\
\overline{\mathrm{Per}_{\mathbf{u}_i}}(\mathsf{f}, \lambda) &:= \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbf{P}(\mathsf{f}(0) \geqslant \lambda, \mathsf{f}(\varepsilon \mathbf{u}_i) < \lambda), i = 1, 2,\\
\overline{\mathrm{Vol}^2}(\mathsf{f}, \lambda) &:= \mathbf{P}(\mathsf{f}(0) \geqslant \lambda),
\end{aligned}$$

and we have, with $\overline{\mathrm{Per}_\infty} = \overline{\mathrm{Per}_{\mathbf{u}_1}} + \overline{\mathrm{Per}_{\mathbf{u}_2}}$,

$$\begin{aligned}
\mathbf{E}[\chi(\mathsf{F} \cap W)] &= \mathrm{Vol}^2(W) \overline{\chi}(\mathsf{f}, \lambda) + \frac{1}{4}(\mathrm{Per}_{\mathbf{u}_2}(W) \overline{\mathrm{Per}_{\mathbf{u}_1}}(\mathsf{f}, \lambda) + \mathrm{Per}_{\mathbf{u}_1}(W) \overline{\mathrm{Per}_{\mathbf{u}_2}}(\mathsf{f}, \lambda))\\
&\qquad + \chi(W) \overline{\mathrm{Vol}^2}(\mathsf{f}, \lambda) \qquad (7.8)\\
\mathbf{E}[\mathrm{Per}_\infty(\mathsf{F} \cap W)] &= \mathrm{Vol}^2(W) \overline{\mathrm{Per}_\infty}(\mathsf{f}, \lambda) + \mathrm{Per}_\infty(W) \overline{\mathrm{Vol}^2}(\mathsf{f}, \lambda) \qquad (7.9)\\
\mathbf{E}[\mathrm{Vol}^2(\mathsf{F} \cap W)] &= \mathrm{Vol}^2(W) \overline{\mathrm{Vol}^2}(\mathsf{f}, \lambda). \qquad (7.10)
\end{aligned}$$

7.4 Gaussian level sets

Let $(\mathsf{f}(x), x \in W)$ be a centred Gaussian field on some $W \in \mathscr{W}_d$. Let the covariance function be defined by

$$\sigma(x, y) = \mathbf{E}[\mathsf{f}(x) \mathsf{f}(y)],\ x, y \in W.$$

Say that some real function h satisfies the Dudley condition on $D \subset W$ if for some $\alpha > 0$, $|h(x) - h(y)| \leqslant |\log(\|x - y\|)|^{-1-\alpha}$ for $x, y \in D$. We will make the following assumption on σ:

Assumption 59. *Assume that* $x \in W \mapsto \partial^2 \sigma(x, x)/\partial x_i \partial y_i$ *exists and satisfies the Dudley condition for* $i = 1, \ldots, d$, *that the partial derivatives* $\partial^4 \sigma(x, x)/\partial_{x_i} \partial_{x_j} \partial_{y_i} \partial_{y_j}, x \in W$, $1 \leqslant i, j \leqslant d$, *exist and that for some finite partition* $\{D_k\}$ *of* W *they satisfy the Dudley condition over each* D_k.

Theorem 60. *Let* $W \in \mathscr{W}_d$ *bounded. Assume that* σ *satisfies Assumption 59 and that for* $x \in W$, $(\mathsf{f}(x), \partial_i \mathsf{f}(x), i = 1, \ldots, d)$ *is non-degenerate. Then for any* $\lambda \in \mathbb{R}$, $\mathsf{F} = F_\lambda(\mathsf{f})$ *satisfies the conclusions of Theorem 56.*

Example 61. Random fields that are $\mathscr{C}^{1,1}$ and not $\mathscr{C}^2$ naturally arise in the context of smooth interpolation. Let $E = \{x_i; i \in \mathbb{Z}\}$ be a countable set of points of $\mathbb{R}$, such that

$x_i < x_{i+1}, i \in \mathbb{Z}$. Let $(W(x), x \in E)$ be a random field on E, and $A_x, B_x, x \in E$ be random variables on the same probability space. Define

$$\mathsf{g}(y) = \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{y\in[x_i,x_{i+1})\}} \left[A_{x_i} \left(\frac{y-x_i}{x_{i+1}-x_i} \right)^2 + B_{x_i} \frac{y-x_i}{x_{i+1}-x_i} + W(x_i) \right].$$

Straightforward computations yield that, with $\Delta_x = W(x_{i+2}) - 2W(x_{i+1}) + W(x_i)$, if

- $A_{x_{i+1}} = \Delta_{x_i} - A_{x_i}, i \in \mathbb{Z}$,
- $B_{x_i} = W(x_{i+1}) - W(x_i) - A_x, i \in \mathbb{Z}$,

then with probability 1, g is a $\mathscr{C}^{1,1}$ and in general not twice differentiable field on $(\lim_{i\to-\infty} x_i, \lim_{i\to\infty} x_i)$ such that $\mathsf{g}(x_i) = W(x_i), i \in \mathbb{Z}$. If for some $i_0 \in \mathbb{Z}$, $(A_{x_{i_0}}; W(x_i), i \in \mathbb{Z})$ is a Gaussian process, g is furthermore a Gaussian process also.

Given a random field $(\mathsf{g}(k); k \in \mathbb{Z}^d)$, it should be possible to carry out a similar approximation scheme in $\mathbb{R}^d$ by defining $\mathsf{g} = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{\{x\in(k+[0,1)^d)\}} \mathsf{g}_k$ where g_k is a bicubic polynomial interpolation of Gaussian variables $W(j), j \in (k + \{0,1\}^d)$ on $k + [0,1)^d$. A possible follow-up of this work could be to investigate the asymptotic properties of topological characteristics of g when it is the smooth interpolation of an irregular Gaussian field as the grid mesh converges to 0.

Let us give the mean Euler characteristic in dimension 2 under the simplifying assumptions that the law of f is invariant under translations and rotations of $\mathbb{R}^2$. This implies for instance that in every $x \in \mathbb{R}^2$, $\mathsf{f}(x), \partial_1 \mathsf{f}(x)$ and $\partial_2 \mathsf{f}(x)$ are independent, see for instance [AT07] Section 5.6 and (5.7.3). Assumption 59 is simpler to state in this context: $x \mapsto \partial^2\sigma(x,x)/\partial x_i \partial y_i$ and $x \mapsto \partial^4 \sigma(x,x)/\partial x_i \partial x_j \partial y_i \partial y_j$ should exist and satisfy Dudley's condition in 0. It actually yields that f has $\mathscr{C}^2$ sample paths, and it is not clear whether this is equivalent to $\mathscr{C}^{1,1}$ regularity in this framework. For this reason we state the result with the abstract conditions of Theorem 58 .

Theorem 62. *Let $\mathsf{f} = (\mathsf{f}(x); x \in \mathbb{R}^2)$ be a $\mathscr{C}^{1,1}$ stationary isotropic centred Gaussian field on $\mathbb{R}^2$ with $\mathbf{E}[\mathrm{Lip}(\partial_i \mathsf{f})^p] < \infty$, for some $p > 6$. Let $\lambda \in \mathbb{R}$, $\mathsf{F} = \{x : \mathsf{f}(x) \geqslant \lambda\}$, and let $W \in \mathscr{W}_2$ bounded. Let $\mu = \mathbf{E}[\partial_1 \mathsf{f}(0)^2]$, and $\Phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_\lambda^\infty \exp(-t^2/2) dt$. Then*

$$\mathbf{E}[\mathrm{Vol}^2(\mathsf{F} \cap W)] = \mathrm{Vol}^2(W)\Phi(\lambda), \tag{7.11}$$

$$\mathbf{E}[\mathrm{Per}_\infty(\mathsf{F} \cap W)] = \mathrm{Vol}^2(W) 2\frac{\sqrt{\mu}}{\pi} \exp(-\lambda^2/2) + \mathrm{Per}_\infty(W)\Phi(\lambda), \tag{7.12}$$

$$\mathbf{E}[\chi(\mathsf{F} \cap W)] = \left(\mathrm{Vol}^2(W) \frac{\mu\lambda}{(2\pi)^{3/2}} + \mathrm{Per}_\infty(W) \frac{\sqrt{\mu}}{4\pi} \right) e^{-\lambda^2/2} + \frac{1}{\sqrt{2\pi}} \Phi(\lambda)\chi(W). \tag{7.13}$$

Remark 63. If W is a square, the relation (7.13) coincides with [AT07, (11.7.14)].

Chapter 8

Realisability as the positive extension of linear operators

The realisability problem is an old general inverse problem about marginals: given some set of parameters ν, is there a random process X which set of marginal parameters is indeed ν? An elementary example is the following: given $a, b \in \mathbb{R}$, is there a random variable X for which a and b are the two first moments? The answer is yes iff $a^2 \leqslant b$. Slightly more complicated, a matrix C of size $n \times n$ is the covariance matrix of a random vector of size n iff it is symmetric semi-definite positive. Similarly, a function $\sigma(x, y)$ on $\mathbb{R}^d$ is the covariance of a random field if it is a semi-definite positive function. In this chapter, we are interested in the question whether $\sigma(x, y)$ is the correlation function of a point process, or if it is the covariance of a random set, which can be seen a $\{0, 1\}$-valued random process.

In general, giving tractable necessary and sufficient condition is out of reach. One of the aims of this section is to uncouple the condition in a combinatorial condition and a regularity condition. To illustrate this fact, let us give the example of a continuous random field on $\mathbb{R}^d$. Is a given function $\sigma(x, y), x, y \in \mathbb{R}^d$, the covariance of a continuous random field? The answer is yes if for any finite set $\{x_1, \ldots, x_q\} \subset \mathbb{R}^d$ and scalar numbers $h_1, \ldots, h_q$,

$$\sum_i \sigma(x_i, x_j) h_i h_j \geqslant 0$$

and if the Dudley criterion is respected: for some $C, \alpha, \eta > 0$,

$$|\sigma(x, x) + \sigma(y, y) - 2\sigma(x, y)| \leqslant C |\log \|x - y\||^{-1-\alpha}, x, y \in \mathbb{R}^d, \|x - y\| < \eta.$$

The latter condition is not necessary though, but is enough to provide the existence of a (Gaussian) field satisfying these conditions. The problem of checking wether σ is "realizable" has been split in two: the semi-positivity, which is of algebraic, or combinatorial nature, and Dudley's criterion, which is about the regularity of σ, and essentially originates from the desired regularity of the field. The general answer we give in the first chapter is a similar abstract uncoupling of the realisability problem.

We will see that such problems are well-posed in terms of the existence of positive extensions of a linear operator on a functional space. It will be applied to give an answer to the realisability problem in the context of point processes in the first chapter. In the second chapter, the same ideas will be used to give answers in the context of random sets.

The remainder of the chapter addresses the existence issue for a rather general random element whose distribution is only partially specified. The technique relies on the existence of a positive extension for linear functionals accompanied by additional conditions that ensure the regularity of the extension needed for interpreting it as a probability measure. It is shown in which case the extension can be chosen to possess some invariance properties.

The results are applied to obtain existence results for point processes with given correlation measure and random closed sets with given two-point covering function (in the next chapter). The regularity conditions ensure that the obtained point processes are indeed locally finite and random sets have regular realisations.

8.1 Introduction

Defining the distribution of a random element ξ in a topological space $\mathbf{S}$ is equivalent to specializing the expected values for all bounded continuous functions $\mathsf{g}(\xi)$. These expected values define a linear functional $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ on the space of bounded continuous functions $\mathsf{g} : \mathbf{S} \mapsto \mathbb{R}$. It is well known that a functional Φ indeed corresponds to a random element if and only if Φ is positive (i.e. $\Phi(\mathsf{g}) \geq 0$ if g is non-negative) and upper semicontinuous (i.e. $\Phi(\mathsf{g}_n) \downarrow 0$ if $\mathsf{g}_n \downarrow 0$), see e.g. [Whi92].

Below we consider the case of functional Φ defined only on some functions on $\mathbf{S}$ and address the *realisability* of Φ, i.e. the mere existence of a random element ξ such that $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ for g from the chosen family $\mathscr{G}$ of functions. The uniqueness is not on the agenda, since typically the family $\mathscr{G}$ will not suffice to uniquely specify the distribution of ξ. A classical example of this setting is the existence of a probability distribution with given marginals, see [Kel64]. The present chapter focuses on some geometric instances of the problem. We will see that in most cases the answer to the existence problem consists of the two main steps.

1. **(Positivity)** Checking the positivity condition on Φ — in most cases this requires checking a system of inequalities, which is a serious (but unavoidable) computational burden.

2. **(Regularity)** Ensuring that the extended functional is regular (namely, upper semicontinuous) and so defines a σ-additive measure.

The first step ensures that it is possible to extend functional Φ positively from a certain family of functions to a wider family. In this work we put the emphasis on the latter step — checking the regularity condition, leaving aside the computational difficulties arising from validating the positivity assumption.

The use of positive extension techniques (that go back to L.V. Kantorovitch) in the framework of stochastic geometry was pioneered by T. Kuna, J. Lebowitz and E.R. Speer [KLS11] in application to point processes, which greatly inspired the current work. In this chapter we establish the general nature of an idea proposed in [KLS11] and show how it leads to various further realisability results. The new idea is to introduce an additional function, what we call the regularity modulus, and to formulate sufficient and necessary conditions in terms of a positive extension of a functional onto the linear space containing the regularity modulus and requiring only a priori integrability of the regularity modulus.

We concentrate on two basic examples of the realisability problem: the existence of point processes with given correlation (factorial moment) measure and the existence of a random closed set with given two-point coverage probabilities or contact distribution functions. The introduction to the realisability issue for point processes is available in several papers by T. Kuna, J. Lebowitz and E.R. Speer [KLS07; KLS11], see also Section 8.3 of this chapter. The realisability problem for random closed sets has been widely studied in physics and material science literature, see [JST07; Mar98; Tor99; Tor06;

TS82] and in particular the comprehensive monograph by S. Torquato [Tor02] and a recent survey by J. Quintanilla [Qui08]. If X is a random closed set in a locally compact Hausdorff second countable space $\mathbb{X}$, its *one-point covering functions* is defined by

$$p_x = \mathbf{P}\{x \in \mathsf{X}\}, \quad x \in \mathbb{X}.$$

It is easy to characterize all one-point covering functions of random closed sets as follows.

Theorem 64. *A function p_x, $x \in \mathbb{X}$, with values in $[0,1]$ is the one-point covering function of a random closed set if and only if p is upper semicontinuous.*

The upper-semi continuity of the one-point covering function of a random closed set ξ is a straightforward consequence of the upper semicontinuity property of the capacity functional of a random closed set, see [Mol05, Sec. 1.1.2]. Conversely, the function p from the theorem is realized (for instance) as the one-point covering function of the random set $\xi = \{x : p_x \geq v\}$ where v is a uniformly distributed variable (the details are left to the reader).

It is considerably more complicated to characterize *two-point covering functions*

$$p_{x,y} = \mathbf{P}\{x, y \in \mathsf{X}\}, \quad x, y \in \mathbb{X}.$$

In view of applications to modeling of random media it is often assumed that X is a stationary set in $\mathbb{R}^d$, so that the one-point covering function is constant and the two-point covering function $p_{x,y}$ depends only on $x - y$. Since a random closed set can be considered as an upper semicontinuous indicator function, the realisability problem for the two-point covering function can be rephrased as follows.

> Characterize covariance functions of (stationary) upper semicontinuous random functions with values in $\{0, 1\}$.

These covariances are obviously a subfamily of positive semi-definite functions. Without the upper semicontinuity requirement, this problem, of combinatorial nature, was solved by B. McMillan [McM55] and L. Shepp [She63; She67] using the extension argument from [Kel64]. More exactly, they normalized indicators by letting them take values $+1$ or -1 and assumed that the mean is zero. Their result does not rely on the topological structure of the underlying space and so does not necessarily lead to an upper semicontinuous indicator function.

Example 65. Let $p_{x,y} = \frac{1}{4}$ and let $p_x = \frac{1}{2}$ for all $x, y \in \mathbb{R}$. While this two-point covering function corresponds, e.g., to the indicator field with independent values, it cannot be obtained as the two-point covering function of a random closed set.

Even leaving aside the upper semicontinuity property, the McMillan–Shepp condition involves a family of corner-positive matrices, which is poorly understood (see [DL97]). As a result, its practical use to check the realisability for random media is rather limited. A number of authors have attempted to come up with simpler (but only necessary) conditions, see, e.g., [JST07; Mat93; Qui08; Tor06]. Another set of conditions for joint distributions of binary random variables is formulated in [SI02] in terms of the corresponding copulas.

The realisability problem can be also posed for point processes in terms of their moment measures. In case of moment measures of arbitrary order it has been solved by A. Lenard [**len75a**; **len75b**]. The case of moment measures up to the second order has been studied by T. Kuna, J. Lebowitz and E.R. Speer [KLS07], whose recent chapter [KLS11] contains (among other results) a *complete* solution of this realisability problem for point processes with finite third-order moments and hard-core type conditions with fixed exclusion distance. The results of [KLS11] can be extended to higher order moment measures, as was explicitly indicated there. Again, the positivity condition of [KLS11] is extremely difficult to verify, even more complicated than the original condition for point processes because of new polynomial functionals involved in the positivity condition.

The chapter is organised as follows. Section 8.2 presents a series of general results on regular extensions and also invariant extensions (relevant for the existence of stationary random elements). These results form the theoretical backbone of our study, and are new even in the abstract setting of extending general positive linear functionals.

Section 8.3 presents a number of realisability conditions for correlation measures of point processes that considerably extend the results of [KLS11] by relaxing the moment and hardcore conditions. One of our most important results is Theorem 78 that shows how to split the positivity and regularity conditions, so that the latter can be efficiently checked. The importance of the packing number in relation to realisability conditions for hard-core point processes is also explained.

8.2 Extending positive functionals

Fundamental results about the extension of positive operators form the heart of our main results, and are necessary to understand the machinery of the proofs. Nevertheless, the results of the subsequent sections can be understood without Section 8.2, with the exception of Definition 70.

8.2.1 General extension theorems

Consider a *vector lattice* $\mathscr{E}$, i.e. a linear space with a partial order and such that for any $\mathsf{v}_1, \mathsf{v}_2 \in \mathscr{E}$, there is an element $\mathsf{v}_1 \vee \mathsf{v}_2 \in \mathscr{E}$, called *maximum* of v_1 and v_2, larger to both v_1 and v_2 and such that any other element larger to both is also larger than $\mathsf{v}_1 \vee \mathsf{v}_2$. The absolute value $|\mathsf{v}|$ of v is defined as the sum of $\mathsf{v} \vee 0$ and $(-\mathsf{v}) \vee 0$.

Let $\mathscr{G}$ be a *vector subspace* of $\mathscr{E}$, which is not necessarily a lattice itself, i.e. $\mathscr{G}$ may be not closed with respect to the maximum operation. We say that $\mathscr{G}$ *majorises* $\mathscr{E}$ if each $\mathsf{v} \in \mathscr{E}$ satisfies $|\mathsf{v}| \leq \mathsf{g}$ for some $\mathsf{g} \in \mathscr{G}$. A real-valued functional Φ defined on $\mathscr{E}$ (resp. $\mathscr{G}$) is said to be *positive* if $\Phi(\mathsf{v}) \geq 0$ whenever $\mathsf{v} \geq 0$ and $\mathsf{v} \in \mathscr{E}$ (resp. $\mathsf{v} \in \mathscr{G}$). A functional defined on $\mathscr{E}$ is said to be an *extension* of $\Phi : \mathscr{G} \mapsto \mathbb{R}$ if it coincides with Φ on $\mathscr{G}$. The extended Φ is always denoted by the same letter. The following result about extension of positive functionals goes back to L.V. Kantorovich.

Theorem 66 (see [AB06], Th. 8.12 and [Vul67], Th. X.3.1)**.** *Assume that $\mathscr{G}$ is a majorising vector subspace of a vector lattice $\mathscr{E}$. Then each positive linear functional on $\mathscr{G}$ admits a positive extension on the whole $\mathscr{E}$.*

If $\mathscr{G}$ is a lattice itself, then it is possible to gain much more control over the extension of Φ, e.g. a continuous functional admits a continuous extension, see [Vul67, Sec. X.5]. On the contrary, very little is known about regularity properties of the extension if $\mathscr{G}$ is not a lattice.

In the following we assume that $\mathscr{G}$ and $\mathscr{E}$ are families of functions on a certain space $\mathbf{S}$. If $\mathscr{G}$ contains constant functions, the positivity of Φ over $\mathscr{G}$ can be equivalently formulated as

$$\Phi(\mathsf{g}) \geq \inf_{x \in \mathbf{S}} \mathsf{g}(X). \tag{8.1}$$

This equivalence is a particular case of the following result for $\chi = 0$ (replace g with $-\mathsf{g}$ in (8.2)).

Proposition 67. *Assume that a vector space $\mathscr{G}$ contains constant functions and denote by $\mathscr{G} \setminus \mathbb{R}$ the family of non-constant functions from $\mathscr{G}$. If χ is any non-negative function on $\mathbf{S}$, then a linear functional Φ on $\mathscr{G}$ admits a positive extension on $\mathscr{G} + \mathbb{R}\chi$ satisfying $\Phi(\chi) = r$ if and only if*

$$r = \sup_{\mathsf{g} \in \mathscr{G},\, \mathsf{g} \le \chi} \Phi(\mathsf{g}) = \sup_{\mathsf{g} \in \mathscr{G} \setminus \mathbb{R}} \inf_{x \in \mathbf{S}} [\chi(x) - \mathsf{g}(x)] + \Phi(\mathsf{g}) < \infty\,. \tag{8.2}$$

Proof. Since every element of $\mathscr{G}$ can be written $c + \mathsf{g}$ with $\mathsf{g} \in \mathscr{G} \setminus \mathbb{R}$ and $c \in \mathbb{R}$, the left-hand side of (8.2) equals

$$r = \sup_{\mathsf{g} \in \mathscr{G} \setminus \mathbb{R}} \sup_{c \in \mathbb{R}:\, c + \mathsf{g} \le \chi} c + \Phi(\mathsf{g}) = \sup_{\mathsf{g} \in \mathscr{G}} c_{\mathsf{g}} + \Phi(\mathsf{g})\,,$$

where $c_{\mathsf{g}} = \inf_{X \in \mathbf{S}} (\chi - \mathsf{g})(x)$ is the largest c such that $c + \mathsf{g} \le \chi$, which yields the equality in (8.2).

The necessity of (8.2) is straightforward because $r \le \Phi(\chi) < \infty$. For the sufficiency, assume that (8.2) holds. The proof consists in checking that assigning the value $\Phi(\chi) = r$ yields a positive extension on $\mathscr{G} + \mathbb{R}\chi$. Let us first prove that Φ is positive on $\mathscr{G}$. If some $\mathsf{g} \le 0$ satisfies $\Phi(\mathsf{g}) > 0$, then $\Phi(t\mathsf{g}) \uparrow \infty$ as $t \to \infty$ whereas $t\mathsf{g} \le \chi$, which contradicts (8.2).

Let $\mathsf{g} + \lambda\chi \ge 0$ for $\lambda \ne 0$ and $\mathsf{g} \in \mathscr{G}$. If $\lambda > 0$, then $-\lambda^{-1}\mathsf{g} \le \chi$, whence $\Phi(-\lambda^{-1}\mathsf{g}) \le r$ and $\Phi(\mathsf{g} + \lambda\chi) \ge -\lambda r + \lambda\Phi(\chi) = 0$. If $\lambda < 0$, $-\lambda^{-1}\mathsf{g} \ge \chi$ whence $-\lambda^{-1}\mathsf{g}$ is larger than any $\mathsf{g}' \le \chi$, and

$$\Phi(-\lambda^{-1}\mathsf{g}) \ge \sup_{\mathsf{g}' \in \mathscr{G}, \mathsf{g}' \le \chi} \Phi(\mathsf{g}') = r$$

by monotonicity of Φ on $\mathscr{G}$. Hence $\Phi(\mathsf{g} + \lambda\chi) \ge -\lambda r + \lambda\Phi(\chi) = 0$. $\square$

The advantage of the latter condition in (8.2) consists in the explicit reference to the space $\mathbf{S}$ where random elements lie instead of checking the inequality $g \le \chi$.

8.2.2 Regularity conditions and distributions of random elements

Let $\mathscr{E}$ be a certain family of functions $\mathsf{v} : \mathbf{S} \mapsto \mathbb{R}$ defined on a space $\mathbf{S}$ with lattice operation being the pointwise maximum and the corresponding partial order.

Theorem 68 (Daniell, see Sec. 4.5 **[dud02]** and Th. 14.1 [Kön97]). *Let a vector lattice $\mathscr{E}$ consist of real-valued functions on $\mathbf{S}$ and let $\mathscr{E}$ contain constants. If Φ is a positive functional on $\mathscr{E}$ such that $\Phi(\mathsf{v}_n) \downarrow 0$ for each sequence $\mathsf{v}_n \downarrow 0$ and $\Phi(1) = 1$, then there exists a unique random element ξ in $\mathbf{S}$, measurable with respect to the σ-algebra generated by all functions from $\mathscr{E}$, such that $\Phi(\mathsf{v}) = \mathbf{E}\mathsf{v}(\xi)$ for all $\mathsf{v} \in \mathscr{E}$.*

In view of the positivity of Φ, the condition imposed on Φ is equivalent to its upper semicontinuity on $\mathscr{E}$. In this chapter, we start with a functional Φ defined on a vector subspace $\mathscr{G} \subset \mathscr{E}$ and discuss the existence of a random element $\xi \in \mathbf{S}$ such that $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi)$ for all $\mathsf{g} \in \mathscr{G}$. In this case Φ is said to be *realisable* as a probability distribution on $\mathbf{S}$.

Assumption 69. *The vector space $\mathscr{G}$ of functions on $\mathbf{S}$ contains constants and, for each $\mathsf{g}_1, \mathsf{g}_2 \in \mathscr{G}$, there exists a $\mathsf{g} \in \mathscr{G}$ such that $(\mathsf{g}_1 \vee \mathsf{g}_2) \le \mathsf{g}$.*

From now on assume that $\mathbf{S}$ is a completely regular topological space, i.e. each closed set and each singleton disjoint from it can be separated by a continuous function.

Definition 70. Given a vector space $\mathscr{G}$ of functions on $\mathbf{S}$, a *regularity modulus* on $\mathbf{S}$ is a lower semicontinuous function $\chi : \mathbf{S} \mapsto [0, \infty]$ such that

$$H_{\mathsf{g}} = \{x \in \mathbf{S} :\ \chi(x) \le \mathsf{g}(x)\} \tag{8.3}$$

is relatively compact for each $\mathsf{g} \in \mathscr{G}$ (if all $\mathsf{g} \in \mathscr{G}$ are bounded, χ is a regularity modulus if and only if it has compact level sets).

Examples of regularity moduli are given in Sections 8.3 and Chapter 9. A measurable function $\mathsf{v} : \mathbf{S} \mapsto \mathbb{R}$ is said to be *χ-regular* if v is continuous on H_{g} for each g in $\mathscr{G}$. Each continuous function is trivially χ-regular. The proof of the following central result is based on the ideas from the proof of [KLS11, Th. 3.14]. It should be noted that our result entails not only the realisability, but also provides a bound for the expected value of the regularity modulus.

Theorem 71. *Consider a vector space $\mathscr{G}$ of functions on $\mathbf{S}$ satisfying Assumption 69 and such that each g from $\mathscr{G}$ is χ-regular for a regularity modulus χ. Let Φ be a linear functional on $\mathscr{G}$ with $\Phi(1) = 1$. Then, for any given $r \geq 0$, there exists a Borel random element ξ in $\mathbf{S}$ such that*

$$\begin{cases} \mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g}) & \text{for all } \mathsf{g} \in \mathscr{G}\,, \\ \mathbf{E}\chi(\xi) \leq r, \end{cases} \tag{8.4}$$

if and only if

$$\sup_{\mathsf{g} \in \mathscr{G},\, \mathsf{g} \leq \chi} \Phi(\mathsf{g}) \leq r\,. \tag{8.5}$$

Proof. Condition (8.5) is necessary because $\mathsf{g} \leq \chi$ implies $\Phi(\mathsf{g}) = \mathbf{E}\mathsf{g}(\xi) \leq \mathbf{E}\chi(\xi) \leq r$.

Sufficiency. Let $\mathscr{E}$ be the family of all χ-regular functions v that satisfy $\mathsf{v} \leq \mathsf{g}$ for some $\mathsf{g} \in \mathscr{G}$. Each function $\mathsf{v} \in \mathscr{E}$ is Borel measurable. Note that $\mathscr{E}$ contains all bounded continuous functions that generate the Baire σ-algebra on $\mathbf{S}$ being in general a sub-σ-algebra of the Borel one. For each $\mathsf{v}_1, \mathsf{v}_2 \in \mathscr{E}$, the function $\mathsf{v}_1 \vee \mathsf{v}_2$ is χ-regular and is majorised by $\mathsf{g}_1 \vee \mathsf{g}_2$, where $\mathsf{g}_1, \mathsf{g}_2 \in \mathscr{G}$ majorise v_1 and v_2 respectively. In view of Assumption 69, $\mathscr{E}$ is a lattice.

Without loss of generality assume that the supremum in (8.5) equals r. By Proposition 67, Φ is positive on $\mathscr{G}$ and can be positively extended onto $\mathscr{G} + \mathbb{R}\chi$ with $\Phi(\chi) = r$, and further on to $\mathscr{E} + \mathbb{R}\chi$ by Theorem 66. It remains to prove that the obtained extension satisfies conditions of Theorem 68. For that, we use an argument similar to that of [KLS11]. First restrict the obtained functional Φ onto $\mathscr{E}$. Assume that χ is strictly positive. Consider a sequence $\{\mathsf{v}_n,\, n \geq 1\} \subset \mathscr{E}$ such that $\mathsf{v}_n \downarrow 0$. For each n, let g_n be a function of $\mathscr{G}$ such that $\mathsf{v}_n \leq \mathsf{g}_n$. Take $\varepsilon > 0$. Then $\mathscr{K}_n = \{x :\ \mathsf{v}_n(x) \geq \varepsilon\chi(x)\}$ is a subset of relatively compact $H_{\mathsf{g}_n/\varepsilon}$, since χ is a regularity modulus. Since v_n is continuous on $H_{\mathsf{g}_n/\varepsilon}$, the set $\mathscr{K}_n$ is closed and therefore compact. The pointwise convergence $\mathsf{v}_n \downarrow 0$ yields that $\cap_n \mathscr{K}_n = \emptyset$ (recall that χ is strictly positive). Since $\{\mathscr{K}_n\}$ is a decreasing sequence of compact sets, $\mathscr{K}_{n_0} = \emptyset$ for some n_0, whence $\mathsf{v}_n(x) < \varepsilon\chi(x)$ for sufficiently large n. The positivity of Φ on $\mathscr{E} + \mathbb{R}\chi$ implies $\Phi(\mathsf{v}_n) \leq \varepsilon\Phi(\chi) = \varepsilon r$, whence $\Phi(\mathsf{v}_n) \downarrow 0$. Theorem 68 yields the existence of a random element ξ in $\mathbf{S}$ such that $\Phi(\mathsf{v}) = \mathbf{E}\mathsf{v}(\xi)$ for all $\mathsf{v} \in \mathscr{E}$.

Since χ is lower semicontinuous and $\mathbf{S}$ is completely regular, it can be pointwisely approximated from below by a sequence $\{\mathsf{v}_n\}$ of non-negative continuous functions, see [Bou66, Ch. 9]. Then $\tilde{\mathsf{v}}_n = \min(n, \mathsf{v}_n)$ belongs to $\mathscr{E}$ and also approximates χ from below, so that $\mathbf{E}\tilde{\mathsf{v}}_n(\xi) = \Phi(\tilde{\mathsf{v}}_n) \leq \Phi(\chi) = r$, while the monotone convergence theorem yields

$$\mathbf{E}\chi(\xi) = \lim_{n \to \infty} \mathbf{E}\tilde{\mathsf{v}}_n(\xi) \leq r\,.$$

If χ is not strictly positive, it suffices to apply the above argument to $\chi' = 1 + \chi$ and use the linearity of Φ. $\square$

Condition (8.5), equivalent to (8.2), is expressed solely in terms of the values taken by Φ on $\mathscr{G}$, and therefore yields a self-contained solution of the realisability problem. It is not easy to check in general, but if χ can be approximated by functions $\chi_n \in \mathscr{G}$, $n \geq 1$, then it is possible to "split" (8.5) into the positivity condition on Φ and the uniform boundedness of $\Phi(\chi_n)$, $n \geq 1$. This idea is used successfully in several different frameworks, which justify the abstract setting of Theorem 71: in Section 8.3 for point processes (see Theorem 76), in Chapter9 for random closed sets (see Theorem 101).

The realisability problem is particularly simple if $\mathbf{S}$ is compact and $\mathscr{G}$ consists of continuous functions. Then, for identically vanishing χ, Theorem 71 yields the following result, which is similar to the Riesz–Markov theorem, see [Kön97].

Corollary 72. *Let $\mathbf{S}$ be a compact space with its Borel σ-algebra. Consider a vector space $\mathscr{G}$ containing constants such that each $\mathsf{g} \in \mathscr{G}$ is continuous and a map $\Phi : \mathscr{G} \mapsto \mathbb{R}$ such that $\Phi(1) = 1$. Then there exists a random element ξ in $\mathbf{S}$ such that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathscr{G}$ if and only if Φ is a linear positive functional on $\mathscr{G}$.*

8.2.3 Passing to the limit

The following result shows that the family of all random elements that realise Φ in the sense of (8.4) is weakly compact.

Theorem 73. *Assume that $\mathscr{G}$ satisfies Assumption 69 and consists of continuous functions on a Polish space $\mathbf{S}$ with regularity modulus χ. Let Φ be a linear positive functional on $\mathscr{G}$. Then the family $\mathscr{M}$ of all Borel random elements ξ that satisfy (8.4) for any given $r \geq 0$ is compact in the weak topology.*

Proof. Since χ is a regularity modulus, the set $H_{r/\varepsilon}$ is compact. By Markov's inequality,

$$\mathbf{P}\{\xi \notin H_{r/\varepsilon}\} = \mathbf{P}\{\chi(\xi) > r/\varepsilon\} \leq \varepsilon\,,$$

for all $\xi \in \mathscr{M}$, so that $\mathscr{M}$ is tight.

Let $\{\xi_n, n \geq 1\}$ be random elements from $\mathscr{M}$. Assume that ξ_n converges weakly to some ξ. Without loss of generality assume that the ξ_n's are defined on the same probability space and converge almost surely to ξ. Since χ is non-negative, Fatou's lemma yields

$$r \geq \liminf \mathbf{E}\chi(\xi_n) \geq \mathbf{E} \liminf \chi(\xi_n) \geq \mathbf{E}\chi(\lim \xi_n) = \mathbf{E}\chi(\xi)\,,$$

where the lower semicontinuity of χ also has been used.

Take an arbitrary $\mathsf{g} \in \mathscr{G}$ and define $H_{\lambda\mathsf{g}}$ as in (8.3). Let $\mathsf{g}^+(x) = \max(\mathsf{g}(x), 0)$ be the positive part of g. Then, for $\lambda > 0$,

$$\mathbf{E}\mathsf{g}^+(\xi_n) = \mathbf{E}\mathsf{g}^+(\xi_n)\,\mathbb{1}_{\xi_n \notin H_{\lambda\mathsf{g}}} + \mathbf{E}\mathsf{g}^+(\xi_n)\,\mathbb{1}_{\xi_n \in H_{\lambda\mathsf{g}}}\ .$$

Since g is continuous, $H_{\lambda\mathsf{g}}$ is closed (and compact), so that if $\xi_n \in H_{\lambda\mathsf{g}}$ for infinitely many n, then also $\xi \in H_{\lambda\mathsf{g}}$. Furthermore, $\lambda\mathsf{g}$ and also g itself, are continuous and bounded on $H_{\lambda\mathsf{g}}$, so that Fatou's lemma yields

$$\begin{aligned}\limsup \mathbf{E}\mathsf{g}^+(\xi_n)\,\mathbb{1}_{\xi_n \in H_{\lambda\mathsf{g}}} &\leq \mathbf{E} \limsup(\mathsf{g}^+(\xi_n)\,\mathbb{1}_{\xi_n \in H_{\lambda\mathsf{g}}}) \\ &\leq \mathbf{E}\mathsf{g}^+(\xi)\,\mathbb{1}_{\xi \in H_{\lambda\mathsf{g}}} \leq \mathbf{E}\mathsf{g}^+(\xi)\,.\end{aligned}$$

Thus

$$\limsup \mathbf{E}\mathsf{g}^+(\xi_n) \leq \mathbf{E}\frac{\chi(\xi_n)}{\lambda} + \mathbf{E}\mathsf{g}^+(\xi) \leq \frac{r}{\lambda} + \mathbf{E}\mathsf{g}^+(\xi)\,.$$

Since λ is arbitrary,

$$\limsup \mathbf{E}\mathsf{g}^+(\xi_n) \leq \mathbf{E}\mathsf{g}^+(\xi)\,.$$

Since g^+ is non-negative, Fatou's lemma yields that $\mathbf{E}\mathsf{g}^+(\xi_n) \to \mathbf{E}\mathsf{g}^+(\xi)$. By applying the same argument to the function $(-\mathsf{g})$, $\lim \mathbf{E}\mathsf{g}(\xi_n) = \mathbf{E}\mathsf{g}(\xi)$, so that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathscr{G}$. Therefore, $\xi \in \mathscr{M}$. □

The following result concerns realisability of pointwise limits of linear functionals. Special conditions of this type for correlation measures of point processes are given in [KLS11, Sec. 3.4].

Theorem 74. *Let $\{\Phi_n,\ n \geq 1\}$ be a sequence of linear positive functionals on a space $\mathscr{G}$ that satisfies the assumptions of Theorem 73. Assume that*

$$\liminf_n \sup_{\mathsf{g}\in\mathscr{G},\, \mathsf{g}\leq\chi} \Phi_n(\mathsf{g}) < \infty\,. \tag{8.6}$$

If $\Phi_n(\mathsf{g}) \to \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathscr{G}$, then Φ is realisable as a random element ξ satisfying (8.4) and such that ξ is the weak limit of random elements realising Φ_{n_k} for a subsequence n_k.

Proof. By passing to a subsequence, it suffices to assume that (8.6) holds for the limit instead of the lower limit. Let ξ_n be a random element that realises Φ_n. If r is larger than the limit of (8.6), then $\mathbf{P}\{\xi_n \notin H_{r/\varepsilon}\} \leq \varepsilon$, so that $\{\xi_n\}$ is a tight sequence. Without loss of generality assume that ξ_n weakly converges to a random element ξ.

The pointwise convergence of Φ_n yields that $\mathbf{E}\mathsf{g}(\xi_n) \to \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathscr{G}$. Now the arguments from the proof of Theorem 73 can be used to show that $\mathbf{E}\mathsf{g}(\xi_n) \to \mathbf{E}\mathsf{g}(\xi)$, so that $\mathbf{E}\mathsf{g}(\xi) = \Phi(\mathsf{g})$ for all $\mathsf{g} \in \mathscr{G}$, i.e. ξ indeed satisfies (8.4). □

8.2.4 Invariant extension

Consider an abelian group Θ of continuous transformations acting on $\mathbf{S}$. For a function v on $\mathbf{S}$, define

$$(\theta\mathsf{v})(x) = \mathsf{v}(\theta x), \quad \theta \in \Theta, x \in \mathbf{S}\,.$$

A functional Φ is said to be Θ-invariant if, for each $\theta \in \Theta$ and v from the domain of definition of Φ, $\Phi(\theta\mathsf{v})$ is defined and equal to $\Phi(\mathsf{v})$.

A Borel random element ξ in $\mathbf{S}$ is said to be Θ-*stationary* if, for each $\theta \in \Theta$, $\theta\xi$ has the same distribution as ξ. A variant of the following result for correlation measures of point processes is given in [KLS11, Th. 4.3].

Theorem 75. *Assume that $\mathscr{G}$ is a Θ-invariant space satisfying Assumption 69 and consisting of χ-regular functions. Furthermore, assume that at least one of the following conditions holds:*

(i) *$\mathscr{G}$ consists of continuous functions and χ is pointwisely approximated from below by a monotone sequence of functions $\mathsf{g}_n \in \mathscr{G}$, $n \geq 1$.*

(ii) *χ is Θ-invariant.*

Let Φ be a Θ-invariant functional on $\mathscr{G}$. Then, for every given $r \geq 0$, there exists a Θ-stationary random element ξ in $\mathbf{S}$ satisfying (8.4) if and only if (8.5) holds.

Proof. **(i)** As in [KLS11, Prop. 4.1], the proof consists in checking hypotheses of the Markov–Kakutani fixed point theorem. Let $\mathscr{M}$ be the family of random elements ξ that realise Φ on $\mathscr{G}$, and satisfy $\mathbf{E}\chi(\theta\xi) \leq r$ for every $\theta \in \Theta$. The family $\mathscr{M}$ is easily seen to be convex with respect to addition of measures, it is compact by Theorem 73, and Θ-invariant, since Φ is Θ-invariant on $\mathscr{G}$. It remains to prove that $\mathscr{M}$ is not empty.

In view of (8.5), it is possible to extend Φ positively onto $\mathscr{G} + \mathbb{R}\chi$, so that $\mathbf{E}\chi(\xi) \leq r$. The Θ-invariance of Φ on $\mathscr{G}$ together with the monotone convergence theorem imply that $\mathbf{E}\chi(\theta\xi) = \mathbf{E}\chi(\xi) \leq r$, whence $\xi \in \mathscr{M}$.

(ii) By Proposition 67, we can extend Φ positively onto the Θ-invariant vector space $\mathscr{V} = \mathscr{G} + \mathbb{R}\chi$. Since Φ is Θ-invariant on $\mathscr{G}$, we have $\Phi(\theta(\mathsf{g}+t\chi)) = \Phi(\theta\mathsf{g}) + t\Phi(\theta\chi) = \Phi(\mathsf{g}+t\chi)$ for $\mathsf{g}+t\chi$ in $\mathscr{V}$, whence Φ is Θ-invariant on $\mathscr{V}$. According to [Sil56, Th. 3], Φ admits a positive Θ-invariant extension to the space $\mathscr{E} + \mathbb{R}\chi$, defined like in the proof of Theorem 71. The restriction of the obtained functional onto $\mathscr{E}$ corresponds to a random element ξ in $\mathbf{S}$ that verifies (8.4) and satisfies $\mathbf{E}(\theta\mathsf{v})(\xi) = \Phi(\theta\mathsf{v}) = \Phi(\mathsf{v}) = \mathbf{E}\mathsf{v}(\xi)$, $\theta \in \Theta$, for v in $\mathscr{E}$. Since $\mathscr{E}$ contains all bounded continuous functions on $\mathbf{S}$, $\theta\xi$ and ξ are identically distributed for all $\theta \in \Theta$. $\square$

8.3 Correlation measures of point processes

8.3.1 Framework and main results

Let $\mathbf{N}$ be the family of locally finite counting measures on a locally compact separable metric space $\mathbb{X}$. We denote the support of $\zeta \in \mathbf{N}$ by the same letter ζ, so that $x \in \zeta$ means $\zeta(\{x\}) \geq 1$.

Equip $\mathbf{N}$ with the vague topology, see [DV88a]. A random element ξ in $\mathbf{N}$ with the corresponding Borel σ-algebra is called a *point process*. Denote by $\mathbf{N}_s$ the family of *simple* counting measures, i.e. those which do not attach mass 2 or more to any given point. If ξ is simple, i.e. $\xi \in \mathbf{N}_s$ a.s., then ξ can be identified with a locally finite random set in $\mathbb{X}$, which is also denoted by ξ.

For a real function h on $\mathbb{X}^2$ and counting measure $\zeta = \sum_i \delta_{x_i}$ given by the sum of Dirac measures, define

$$\mathsf{g}_h(\zeta) = \sum_{x_i, x_j \in \zeta,\ i \neq j} h(x_i, x_j)\,,$$

whenever the series absolutely converges, the empty sum being 0. Note that the sum in the right-hand side is taken over all pairs of distinct points from the support of ζ, where multiple points appear several times according to their multiplicities. The value $\mathsf{g}_h(\zeta)$ is necessarily finite if h is bounded and has a bounded support. The value $\mathsf{g}_h(\zeta)$ is termed in [KLS11] the quadratic polynomial of ζ, while polynomials of order $n \geq 1$ are sums of functions of n points of the process, and are constants if $n = 0$.

Let $\mathscr{G}$ be the vector space formed by constants and functions g_h for h from the space $\mathscr{C}_c$ of symmetric continuous functions with compact support. Note that $\mathscr{G}$ satisfies Assumption 69, since

$$(c_1 + \mathsf{g}_{h_1}) \vee (c_2 + \mathsf{g}_{h_2}) \leq c_1 \vee c_2 + \mathsf{g}_{h_1 \vee h_2} \in \mathscr{G}$$

for all $c_1, c_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathscr{C}_c$. Furthermore, each g_h is continuous in the vague topology, and so is χ-regular for any regularity modulus χ.

Assume that ξ has locally finite second moment, i.e. $\mathbf{E}\xi(A)^2$ is finite for each bounded A. The *correlation measure* ρ (also called the second factorial moment measure) of a point process ξ is a measure on $\mathbb{X} \times \mathbb{X}$ that satisfies

$$\int_{\mathbb{X}\times\mathbb{X}} h(x,y)\rho(dxdy) = \mathbf{E}\mathsf{g}_h(\xi) \tag{8.7}$$

for each $h \in \mathscr{C}_c$, see [DV88a, Sec. 5.4] and [SKM95, Sec. 4.3]. The left-hand side defines a linear functional $\Phi(\mathsf{g}_h)$ on $\mathsf{g}_h \in \mathscr{G}$.

Let $\mathbf{N}_0$ be a subset of $\mathbf{N}$, which may be $\mathbf{N}$ itself. Given a symmetric locally finite measure ρ on $\mathbb{X} \times \mathbb{X}$, the *realisability problem* amounts to the existence of a point process ξ with realisations from $\mathbf{N}_0$ and with correlation measure ρ, so that $\Phi(\mathsf{g}_h) = \mathbf{E}\mathsf{g}_h(\xi)$ for all $h \in \mathscr{C}_c$.

By (8.1), the positivity of Φ means

$$\Phi(\mathrm{g}_h) \geq \inf_{Y \in \mathbf{N}_0} \mathrm{g}_h(Y) \tag{8.8}$$

for all $h \in \mathscr{C}_c$. Then it is clear that the positivity of Φ is necessary for its realisability. If $\mathbf{N}$ is compact in the vague topology, then Corollary 72 applies and the positivity condition (8.8) is necessary and sufficient for the realisability of ρ.

However, in general the positivity condition alone is not sufficient for the realisability, see [KLS07, Ex. 3.12]. In the following we find another condition that is not directly related to the positivity, but, together with the positivity, is necessary and sufficient for the realisability.

As an introduction, let us present our results for $\mathbb{X}$ being a subset of the Euclidean space $\mathbb{R}^d$. For $\varepsilon \geq 0$, define

$$\chi_\varepsilon(\zeta) = \sum_{x,y \in \zeta,\, x \neq y} \|x-y\|^{-d-\varepsilon}, \quad \zeta \in \mathbf{N}\,,$$

which is later acknowledged as being a regularity modulus (see Definition 70) if $\varepsilon \neq 0$. Note that $\chi_\varepsilon(\zeta)$ is infinite if ζ has multiple points. The tools developed in this chapter enable us to resolve the original realisability problem with a supplementary regularity condition involving χ_ε.

Theorem 76. ***(i)*** *Let $\mathbb{X}$ be a compact subset of $\mathbb{R}^d$ without isolated points. A symmetric finite measure $\rho(dxdy)$ on $\mathbb{X}^2$ is the correlation measure of a simple point process $\xi \subset \mathbb{X}$ such that $\mathbf{E}\chi_0(\xi) < \infty$ if and only if Φ given by the left-hand side of (8.7) is positive and*

$$\int_{\mathbb{X}^2} \|x-y\|^{-d} \rho(dxdy) < \infty\,.$$

(ii) *Let ρ be a symmetric locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\rho((A+x)\times(B+x)) = \rho(A \times B)$ for all $x \in \mathbb{R}^d$ and measurable sets A and B. Then there exists a simple stationary point process ξ with correlation measure ρ, such that*

$$\mathbf{E}\chi_0(\xi \cap C) < \infty$$

for every compact $C \subset \mathbb{R}^d$, if and only if Φ defined by (8.7) is positive and

$$\int_{B \times B} \|x-y\|^{-d} \rho(dxdy) < \infty \tag{8.9}$$

for some open set B.

The proof can be found in [LM15]. The first statement follows from Theorem 80 using the fact that the packing number P_t of $\mathbb{X}$ is bounded by ct^{-d} for all sufficiently small t. For **(ii)**, apply Theorem 84**(ii)** noticing that the imposed condition is equivalent to (8.24).

In the following subsections, one can find a quantification of this result (i.e. how the left hand member of (8.9) controls the value of $\mathbf{E}\chi_0(X \cap C)$) as well as generalisations for general metric spaces. The main argument used is a splitting method based on Theorem 71, the details are made clear in the proof of Theorem 78. Note that the packing number of the metric space appears as a crucial quantity to uncouple in this way the realisability problem, see Lemma 77.

8.3.2 Moment conditions

The family $\mathbf{N}_k$ of all counting measures with total mass at most k on a compact space $\mathbb{X}$ is compact. Thus, a measure ρ on $\mathbb{X}^2$ is realisable as a point process with at most k points if (8.8) holds with $\mathbf{N} = \mathbf{N}_k$.

Assume that ζ is a finite counting measure. For $\alpha > 2$ define

$$\chi_\alpha(\zeta) = \zeta(\mathbb{X})^\alpha, \quad \zeta \in \mathbf{N}.$$

The finiteness of $\mathbf{E}\chi_\alpha(\xi)$ amounts to the finiteness of the moment of order α for the total mass of ξ. Since $h \in \mathscr{C}_c$ is bounded by a constant c' and $\alpha > 2$, the family

$$\{\zeta \in \mathbf{N} :\ \chi_\alpha(\zeta) \le c + \mathsf{g}_h(\zeta)\} \subset \{\zeta \in \mathbf{N} :\ \zeta(\mathbb{X})^\alpha \le c + c'\zeta(\mathbb{X})^2\}$$

consists of counting measures with total masses bounded by a certain constant and therefore is compact in the space $\mathbf{N}$. Hence χ_α is a regularity modulus and so Theorem 71 yields the realisability condition

$$\sup_{\mathsf{g} \in \mathscr{G},\ \mathsf{g} \le \chi_\alpha} \Phi(\mathsf{g}) < \infty \tag{8.10}$$

of ρ by a point process ξ whose total number of points has finite moment of order α. Note that [KLS11, Th. 3.14] provides a variant of this result assuming the existence of the third factorial moment of the cardinality of ξ (i.e. with $\alpha = 3$) and for the joint realisability of the intensity and the correlation measures. The condition of [KLS11, Th. 3.14] (reformulated for the correlation measure only) reads in our notation as $c + \Phi(\mathsf{g}_h) + br \ge 0$ whenever $c + \mathsf{g}_h + b\chi_3$ is non-negative on $\mathbf{N}$. Noticing that $b \ge 0$, this is equivalent to the fact that $c + \Phi(\mathsf{g}_h) \le r$ whenever $c + \mathsf{g}_h \le \chi_3$, being exactly (8.10). If Θ is a group of continuous transformations acting on $\mathbb{X}$ and ρ is Θ-invariant, then the point process ξ can be chosen Θ-stationary by Theorem 75(ii).

In order to handle possibly non-finite point processes ξ define

$$\chi_{\alpha,\beta}(\zeta) = \left(\sum_{x \in \zeta} \beta(x)\right)^\alpha, \quad \zeta \in \mathbf{N},$$

for a lower semicontinuous strictly positive function $\beta : \mathbb{X} \mapsto \mathbb{R}$ and $\alpha > 2$. By approximating β from below with compactly supported functions, it is easy to see that $\chi_{\alpha,\beta}$ is a regularity modulus. By Theorem 71 and (8.2), for any given $r \ge 0$, there is a point process ξ with correlation measure ρ such that $\mathbf{E}\chi_{\alpha,\beta}(\xi) \le r$ if and only if ρ satisfies

$$\inf_{\zeta \in \mathbf{N}} [\chi_{\alpha,\beta}(\zeta) - \mathsf{g}_h(\zeta)] + \int_{\mathbb{X}^2} h(x,y)\rho(dxdy) \le r, \quad h \in \mathscr{C}_c. \tag{8.11}$$

For $\alpha = 3$, condition (8.11) is a reformulation of [KLS11, Th. 3.17] meaning the positivity of Φ on a family of positive polynomials that involve symmetric functions of the support points up to the third order. The realisability condition for Θ-stationary random elements can be obtained by applying Theorem 75.

8.3.3 Hardcore point processes on a compact space

Assume that $\mathbb{X}$ is a compact metric space with metric d. Let $\mathbf{N}_\varepsilon$ be the family of ε*-hardcore* point sets in $\mathbb{X}$ (including the empty set), i.e. each $\zeta \in \mathbf{N}_\varepsilon$ attaches unit masses to distinct points with pairwise distances at least ε with a fixed $\varepsilon > 0$. In this case no multiple points are allowed, i.e. $\mathbf{N}_\varepsilon \subset \mathbf{N}_s$.

According to [HS78; KK06], a subset $\mathbf{N}$ of simple counting measures $\mathbf{N}_s$ is relatively compact if and only if $\sup\{\zeta(K) : \zeta \in \mathbf{N}\}$ is finite and the infimum over $\zeta \in \mathbf{N}$ of the minimal distance between two points in $\zeta \cap K$ is strictly positive for each compact set $K \subset \mathbb{X}$. The hard-core condition yields that the number of points in any compact set is uniformly bounded, and so $\mathbf{N}_\varepsilon$ is indeed compact. By Corollary 72, ρ is realisable as the correlation measure of an ε-hard-core point process with given $\varepsilon > 0$ if and only if

$$\Phi(\mathsf{g}_h) \geq \inf_{\zeta \in \mathbf{N}_\varepsilon} \mathsf{g}_h(\zeta) \tag{8.12}$$

for all $h \in \mathscr{C}_c$. This result is formulated in [KLS11, Th. 3.4], which essentially reduces to the positivity of Φ over the family $c + \mathsf{g}_h$ (in our setting).

In this chapter we assume that the hardcore distance is not predetermined and the point process takes realisations from $\cup_{\varepsilon>0}\mathbf{N}_\varepsilon$, which coincides with $\mathbf{N}_s$ in case of compact $\mathbf{N}$. Note that (8.12) is stronger than the positivity of Φ on functions g_h defined on the whole family $\mathbf{N}_s$ and formulated as

$$\Phi(\mathsf{g}_h) \geq \inf_{\zeta \in \mathbf{N}_s} \mathsf{g}_h(\zeta)\,, \quad h \in \mathscr{C}_c. \tag{8.13}$$

If $\mathbb{X}$ does not have isolated points, then the infimum in (8.13) can be taken over $\mathbf{N}$. This is seen by approximating a multiple atom with a sequence of simple counting measures supported by points converging to the atom's location.

In the following we use the (hard-core) regularity modulus of the form

$$\chi_\psi^{\mathrm{hc}}(\zeta) = \sum_{x_i, x_j \in \zeta,\ i \neq j} \psi(\mathsf{d}(x_i, x_j)), \quad \zeta \in \mathbf{N}_s\,,$$

where $\psi : (0,\infty) \mapsto [0,\infty]$ is a monotone decreasing right-continuous function, such that $\psi(t) \to \infty$ as $t \downarrow 0$. The compactness of $\mathbb{X}$ and the lower semicontinuity of ψ imply that χ_ψ^{hc} is lower semicontinuous on $\mathbf{N}_s$. As shown below χ_ψ^{hc} is a regularity modulus if ψ grows sufficiently fast at zero.

Let P_t be the *packing number* of $\mathbb{X}$, i.e. the maximum number of points in $\mathbb{X}$ with pairwise distances exceeding t, see [Mat95, p. 78]. It is convenient to define the packing number at $t = 0$ as $\mathsf{P}_0 = \infty$ if $\mathbb{X}$ is infinite and otherwise let P_0 be the cardinality of $\mathbb{X}$. The following lemma is proved in [LM15]:

Lemma 77. *Function χ_ψ^{hc} is a regularity modulus on $\mathbf{N}_s$ if*

$$\psi(t)/\mathsf{P}_t \to \infty \qquad \text{as } t \downarrow 0\,. \tag{8.14}$$

The following theorem shows that the realisability condition can be split into the positivity condition (8.13) on the linear functional Φ and the regularity condition (8.15) on the correlation measure, so that the latter can be easily checked. Such a split is possible because the regularity modulus χ_ψ^{hc} can be approximated by functions from $\mathscr{G}$.

Theorem 78. *A locally finite measure ρ on $\mathbb{X}^2$ is the correlation measure of a simple point process ξ such that $\mathbf{E}\chi_\psi^{\mathrm{hc}}(\xi) \leq r$ for some $r \geq 0$ with ψ satisfying (8.14) if and only if (8.13) holds and*

$$\int_{\mathbb{X}^2} \psi(\mathsf{d}(x,y))\rho(dxdy) \leq r\,. \tag{8.15}$$

The following result is obtained by letting ψ be infinite on $[0,\varepsilon)$ and otherwise setting it to zero.

Corollary 79. *A measure ρ on $\mathbb{X}^2$ is the correlation measure of a point process ξ with $\xi \in \mathbf{N}_\varepsilon$ a.s. if and only if (8.13) holds and $\rho(\{(x,y) : \mathsf{d}(x,y) < \varepsilon\}) = 0$.*

The following result yields a direct realisability condition for ρ without mentioning a regularity modulus.

Theorem 80. *Let ρ be a locally finite measure on $\mathbb{X}^2$, and fix any $r \geq 0$. Then there exists, for every $r' > r$, a simple point process ξ with correlation measure ρ, such that*

$$\mathbf{E} \sum_{x_i, x_j \in \xi,\ i \neq j} \mathsf{P}_{\mathsf{d}(x_i, x_j)} \leq r' , \tag{8.16}$$

if and only if (8.13) holds and

$$\int_{\mathbb{X}^2} \mathsf{P}_{\mathsf{d}(x,y)} \rho(dxdy) \leq r. \tag{8.17}$$

Remark 81. Let Θ be a group of continuous transformations on $\mathbb{X}$ that leave ρ invariant, i.e. $\rho(\theta A \times \theta B) = \rho(A \times B)$ for all $\theta \in \Theta$ and Borel A, B. Since the regularity modulus χ_ψ^{hc} can be approximated from below by a sequence of functions from $\mathscr{G}$, Theorem 75(i) is applicable and so the corresponding point process ξ in Theorems 78, 80 and Corollary 79 can be chosen Θ-stationary. If Θ consists of isometric transformations, then Theorem 75(ii) is also applicable.

8.3.4 Non-compact case and stationarity

Assume that $\mathbb{X} = \mathbb{R}^d$ and $\mathsf{d}(x, y) = \|x - y\|$ is the Euclidean metric. Let ψ be a positive right-continuous monotone function on $\mathbb{R}_+$ such that $\psi(t)t^d \to \infty$ as $t \to 0$. Denote by B_n the open ball of radius n centred at 0. Given a known bound for the packing number in the Euclidean space [Mat95, p. 78], Lemma 77 implies that χ_ψ^{hc} is a regularity modulus on every B_n, $n \geq 1$. Define

$$\chi_{\beta\psi}^{\mathrm{hc}}(Y) = \sum_{x_i, x_j \in Y,\ i \neq j} \beta(x_i, x_j) \psi(\|x_i - x_j\|) \tag{8.18}$$

for a bounded lower semicontinuous strictly positive on $\mathbb{R}^d \times \mathbb{R}^d$ function β.

Theorem 82. *Let ρ be a locally finite measure on $\mathbb{R}^d \times \mathbb{R}^d$.*

(i) *The measure ρ is realisable as the correlation measure of a point process ξ that satisfies $\mathbf{E}\chi_{\beta\psi}^{\mathrm{hc}}(\xi) \leq r$ if and only if (8.13) holds and*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \beta(x, y) \psi(\|x - y\|) \rho(dxdy) \leq r. \tag{8.19}$$

(ii) *Fix $r \geq 0$, let*

$$r_n = \int_{B_n \times B_n} \|x - y\|^{-d} \rho(dxdy), \quad n \geq 1, \tag{8.20}$$

and let $\{\beta_n,\ n \geq 1\}$ be a sequence of non-increasing numbers converging to 0. Then the following assertions are equivalent.

(a) (8.13) holds and

$$\sum_{n \geq 1} \beta_n (r_{n+1} - r_n) \leq r < \infty, \tag{8.21}$$

in particular every r_n, $n \geq 1$, is finite.

(b) For every $r' > r$ there exists ξ with correlation measure ρ and such that

$$\sum_{n\geq 1}(\beta_n - \beta_{n+1})\mathbf{E} \sum_{x_i, x_j \in B_n,\, i\neq j} \|x_i - x_j\|^{-d} \leq r' . \tag{8.22}$$

Remark 83. Remark for point **(ii)** that if each r_n, $n \geq 1$, is finite, there always exists a sequence $\{\beta_n\}$ of sufficiently small numbers such that the right-hand side of (8.21) is finite.

If the distribution of point process ξ is invariant with respect to the group Θ of translations of $\mathbb{R}^d$, then ξ is called *stationary*. Its correlation measure ρ is translation invariant, i.e. $\rho((A+x)\times(B+x)) = \rho(A\times B)$ for all $x \in \mathbb{R}^d$ and so

$$\rho(A\times B) = \lambda^2 \int_A \int_{\mathbb{R}^d} \mathbb{1}_{x+y\in B}\, \bar{\rho}(dy)dx\,, \tag{8.23}$$

where λ is the intensity of ξ and $\bar{\rho}$ is a measure on $\mathbb{R}^d$ called the *reduced* correlation measure, see [SW08, p. 76].

Theorem 84. *Let $\bar{\rho}$ be a locally finite measure on $\mathbb{R}^d$, let β be a bounded lower semicontinuous strictly positive function on $\mathbb{R}^d$ satisfying*

$$\bar{\beta}(y) = \int_{\mathbb{R}^d} \beta(x, x+y)dx < \infty\,, \quad y \in \mathbb{R}^d\,,$$

and let ψ be a monotone decreasing non-negative function such that $t^d\psi(t) \to \infty$.

(i) *$\bar{\rho}$ is the reduced correlation measure of a stationary point process ξ that satisfies $\mathbf{E}\chi^{\mathrm{hc}}_{\beta\psi}(\xi) \leq r$ if and only if (8.13) holds and*

$$\int_{\mathbb{R}^d} \bar{\beta}(y)\psi(\|y\|)\bar{\rho}(dy) \leq r\,.$$

(ii) *$\bar{\rho}$ is realisable as the reduced correlation measure of a stationary point process ξ that satisfies (8.22) for some sequence $\{\beta_n,\, n \geq 1\}$ if and only if*

$$\int_B \|y\|^{-d}\bar{\rho}(dy) < \infty \tag{8.24}$$

for some open ball B containing the origin. If $\int_{\mathbb{R}^d} \|y\|^{-d}\bar{\rho}(dy)$ is finite, it is possible to let $\beta_n = n^{-d-\delta}$, $n \geq 1$, for any fixed $\delta > 0$.

Proof. It suffices to use (8.23) to confirm the conditions imposed in Theorem 82, see also Remark 83. In order to show that ξ can be chosen stationary, note that $\chi^{\mathrm{hc}}_{\beta\psi}$ can be pointwisely approximated from below by a monotone sequence of functions from $\mathscr{G}$, so Theorem 75(i) applies. $\square$

8.3.5 Joint realisability of the intensity and correlation

Recall that the intensity measure ρ_1 of a point process ξ is defined from

$$\mathbf{E}\sum_{x_i\in\xi} h(x_i) = \int h(x)\rho_1(dx)\,, \quad h \in \mathscr{C}^1_c\,,$$

where $\mathscr{C}^1_c$ is the family of continuous functions on $\mathbb{X}$ with compact support. A pair (ρ_1, ρ) of locally finite non-negative measures on $\mathbb{X}$ and $\mathbb{X}^2$ respectively is said to be jointly realisable if there exists a point process ξ with intensity measure ρ_1 and correlation measure ρ.

Let $\mathscr{G}_1$ be the vector space formed by constants and functions

$$\mathsf{g}_{h_1,h}(Y) = \sum_{x \in Y} h(x) + \mathsf{g}_h(Y), \quad Y \in \mathbf{N},$$

for $h_1 \in \mathscr{C}_c^1$ and $h \in \mathscr{C}_c$. It is easy to see that Assumption 69 is verified in this case. The pair (ρ_1, ρ) yields a linear functional

$$\Phi(\mathsf{g}_{h_1,h}) = \int_{\mathbb{X}} h_1(x)\rho_1(dx) + \int_{\mathbb{X}^2} h(x,y)\rho(dxdy). \tag{8.25}$$

The realisability of Φ by a point process ξ means that $\Phi(\mathsf{g}_{h_1,h}) = \mathbf{E}\mathsf{g}_{h_1,h}(\xi)$. Functional Φ is positive on $\mathscr{G}_1$ if and only if

$$\Phi(\mathsf{g}_{h_1,h}) \ge \inf_{Y \in \mathbf{N}} \mathsf{g}_{h_1,h}(Y), \quad h_1 \in \mathscr{C}_c^1,\ h \in \mathscr{C}_c. \tag{8.26}$$

Similar arguments as before apply and yield the joint realisability conditions. Consider the special case of stationary processes in $\mathbb{X} = \mathbb{R}^d$ with the reduced correlation measure $\bar\rho$ (see (8.23)) and intensity $\rho_1(dx) = \lambda dx$ proportional to the Lebesgue measure.

Theorem 85. *Let λ be a constant, and let $\bar\rho$ be a locally finite measure of $\mathbb{R}^d$. Then there is a stationary point process ξ with intensity $\rho_1(dx) = \lambda dx$ and reduced correlation measure $\bar\rho$ if Φ given by (8.25) satisfies (8.26) with $\mathbf{N} = \mathbf{N}_s$ and*

$$\int_B \|z\|^{-d}\bar\rho(dz) < \infty$$

for some open set B containing the origin.

Proof. It suffices to note that $\mathsf{g}_{h_1,h}$ is dominated by $c\mathsf{g}_h$ for a constant c and follow the proof of **(ii)** in Theorem 82. The condition on $\bar\rho$ follows from (8.20) and (8.23). □

Chapter 9

Realisability of random sets

We provide a characterization of realisable set covariograms, bringing a rigorous yet abstract solution to the S_2 problem in materials science. Our method is based on the covariogram functional for random measurable sets (RAMS) and on a result about the representation of positive operators on a non-compact space. RAMS are an alternative to the classical random closed sets in stochastic geometry and geostatistics, they provide a weaker framework allowing to manipulate more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of geometric nature. Along the way, we extend the theory of random measurable sets, and in particular the local approximation of the perimeter by local covariograms.

9.1 Framework and main results

9.1.1 Introduction

An old and difficult problem in materials science is the S_2 problem, often posed in the following terms: Given a real function $S_2 : \mathbb{R}^d \to [0,1]$, is there a stationary random set $\mathsf{X} \subset \mathbb{R}^d$ whose standard two point correlation function is S_2, that is, such that

$$\mathbf{P}(x, y \in \mathsf{X}) = S_2(x-y),\ x, y \in \mathbb{R}^d \ ? \tag{9.1}$$

The S_2 problem is a *realisability problem* concerned with the existence of a (translation invariant) probability measure satisfying some prescribed marginal conditions.

This question is the stationary version of the problem of characterizing functions $S(x,y)$ satisfying

$$S(x,y) = \mathbf{P}(x, y \in \mathsf{X}) = \mathbf{E}\mathbf{1}_{\mathsf{X}}(x)\mathbf{1}_{\mathsf{X}}(y).$$

The right-hand term is the second order moment of the random indicator field $x \mapsto \mathbf{1}_{\mathsf{X}}(x)$, which justifies the term of *realisability problems*, concerned with the existence of a positive measure satisfying some prescribed moment conditions. See Section 8.1 for more background on this problem.

One can see the S_2 problem as a truncated version of the general *moment problem* that deals with the existence of a process for which all moments are prescribed. The main difficulty in only considering the moments up to some finite order is that this sequence of moments does not uniquely determine the possible solution. The appearance of second order realisability problems for random sets goes back to the 1950's, see for instance [McM55] in the field of telecommunications. There are applications in materials science and geostatistics, and marginal problems in general are present under different occurrences in fields as various as quantum mechanics, computer science, or game theory, see the recent work [FC13] and references therein.

Reconstruction of heterogeneous materials from a knowledge of limited microstructural information (a set of lower-order correlation functions) is a crucial issue in many applications. Finding a constructive solution to the realisability problem described above should allow one to test whether an estimated covariance indeed corresponds to a random structure, and propose an adapted reconstruction procedure. Studying this problem can serve many other purposes, especially in spatial modelling, where one needs to know necessary admissibility conditions to propose new covariance models. A series of works by Torquato and his coauthors in the field of materials science gathers known necessary conditions and illustrate them for many 2D and 3D theoretical models, along with reconstruction procedures (see [JST07] and the survey [**Tor02**] and references therein). This question was developed alongside in the field of geostatistics, where some authors do not tackle directly this issue, but address the realisability problem within some particular classes of models, e.g. Gaussian, mosaic, or Boolean model (see [Mas72; CD99; Lan02; Eme10]).

A related question concerns the *specific covariogram* of a stationary random set X, defined for all non empty bounded open sets $U \subset \mathbb{R}^d$ by

$$\gamma^s_{\mathsf{X}}(y) = \frac{\mathbf{E}\mathcal{L}(\mathsf{X} \cap (y+\mathsf{X}) \cap U)}{\mathcal{L}(U)} = \mathbf{E}\mathcal{L}(\mathsf{X} \cap (y+\mathsf{X}) \cap (0,1)^d), \tag{9.2}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^d$. The associated realisability problem, which consists in determining whether there exists a stationary random set X whose specific covariogram is a given function, is the *(specific) covariogram realisability problem.* Note that a straightforward Fubini argument gives that for any stationary random closed set X

$$\gamma^s_{\mathsf{X}}(y) = \int_{(0,1)^d} \mathbf{P}(x \in \mathsf{X}, x-y \in \mathsf{X})dx = S_2(-y) = S_2(y), \tag{9.3}$$

and thus the S_2 realisability problem and the specific covariogram problem are fundamentally the same.

Our main result provides an abstract and fully rigorous characterization of this problem for random measurable sets (RAMS) having locally finite mean perimeter. Furthermore, in the restrictive one-dimensional case ($d = 1$), results can be passed on to the classical framework of random closed sets. It will become clear in this chapter why the covariogram approach in the framework of random measurable sets is more adapted to a rigorous mathematical study. Random measurable sets are an alternative to the classical random closed sets in stochastic geometry and geostatistics, they provide a weaker framework allowing to manipulate more irregular functionals, such as the perimeter. We therefore use the illustration provided by the S_2 problem to advocate the use of RAMS for solving theoretical problems of geometric nature. Along the way, we extend the theory of random measurable sets, and in particular the local approximation of the perimeter by local covariograms. Let us remark that the framework of RAMS is related to the one of "random sets of finite perimeter" proposed recently by Rataj [Rat14]. However it is less restrictive since RAMS do not necessarily have finite perimeter.

Our main result uses a fundamental relation between the Lipschitz property of the covariogram function of a random set, and the finiteness of its mean variational perimeter, unveiled in [Gal11]. Like in the previous chapter about point processes, we prove that the realisability of a given function $S_2 : \mathbb{R}^d \to \mathbb{R}$ can be characterized by two independent conditions : a positivity condition, and a regularity condition, namely the Lipschitz property of S_2. The positivity condition deals with the positivity of a linear operator extending S_2 on an appropriate space, and is of combinatorial nature. The proof of this main result relies on a theorem dealing with positive operators on a non-compact space recently derived in [LM15] to treat realisability problems for point processes. This general method therefore proves here its versatility by being applied in the framework of random sets in a very similar manner.

Checking whether S_2 satisfies the positivity condition is completely distinct from the concerns of this chapter. It is a difficult problem that has a long history. It is more or less implicit in many articles, and has been, to the best of the authors' knowledge, first addressed directly by Shepp [She63], later on by Matheron [Mat93], and more recently in [Qui08],[Lac18b]. It is equivalent to the study of the *correlation polytope* in the discrete geometry literature, see for instance the works of Deza and Laurent [DL97]. Still, a deep mathematical understanding of the problem remains out of reach.

The plan of the chapter is as follows. We give in the subsections below a quick overview of the mathematical objects involved here, namely random measurable sets, positivity, perimeter, and realisability problems, and we also state the main result of the chapter dealing with the specific covariogram realisability problem for stationary random measurable sets with finite specific perimeter. In Section 9.2, we develop the theory of random measurable sets, define different notions of perimeter, and explore the relations with random closed sets. In Section 9.3, we give the precise statement and the proof of the main result. We also show that our main result extends to the framework of one-dimensional stationary RACS.

9.1.2 Random measurable sets and variational perimeter

Details about random measurable sets are presented in Section 9.2, and we give here the essential notation for stating the results. Call $\mathscr{B}_d$ the class of Lebesgue measurable sets of $\mathbb{R}^d$. A *random measurable set* (RAMS) X is a random variable taking values in $\mathscr{B}_d$ endowed with the Borel σ-algebra induced by the local convergence in measure (which corresponds to the $\mathscr{L}^1_{\text{loc}}(\mathbb{R}^d)$-topology for the indicator functions, see Section 9.2.1 for details). Remark that under this topology, one is bound to identify two sets A and B lying within the same Lebesgue class (that is, such that their symmetric difference $A\Delta B$ is Lebesgue-negligible), and we indeed perform this identification on $\mathscr{B}_d$. Say furthermore that a RAMS is *stationary* if its law is invariant under translations of $\mathbb{R}^d$.

One geometric notion that can be extended to RAMS is that of perimeter. For a deterministic measurable set A, the perimeter of A in an open set $U \subset \mathbb{R}^d$ is defined as the variation of the indicator function $\mathbf{1}_A$ in U, that is,

$$\operatorname{Per}(A;U) = \sup\left\{ \int_U \mathbf{1}_A(x)\operatorname{div}\varphi(x)dx :\ \varphi \in \mathscr{C}^1_c(U,\mathbb{R}^d),\ \|\varphi(x)\|_2 \le 1 \text{ for all } x \right\}, \quad (9.4)$$

where $\mathscr{C}^1_c\left(U,\mathbb{R}^d\right)$ denotes the set of continuously differentiable functions $\varphi : U \to \mathbb{R}^d$ with compact support and $\|\cdot\|_2$ is the Euclidean norm [AFP00] (See Section 9.2.2 for a discussion and some properties of variational perimeters). If X is a RAMS then for all open sets $U \subset \mathbb{R}^d$, $\operatorname{Per}(\mathsf{X};U)$ is a well-defined random variable (since the map $A \mapsto \operatorname{Per}(A;U)$ is lower semi-continuous for the local convergence in measure in $\mathbb{R}^d$ [AFP00, Proposition 3.38]). Besides, if X is stationary then $U \mapsto \mathbf{E}(\operatorname{Per}(\mathsf{X};U))$ extends into a measure invariant by translation, and thus proportional to the Lebesgue measure. One calls *specific perimeter* or (*specific variation* [Gal14]) of X the constant of proportionality that will be denoted by $\overline{\operatorname{Per}}(\mathsf{X})$ and that is given by $\overline{\operatorname{Per}}(\mathsf{X}) = \mathbf{E}\operatorname{Per}(\mathsf{X};(0,1)^d)$. We refer to [Gal11] for the computation of the specific perimeter of some classical random set models (Boolean models and Gaussian level sets).

9.1.3 Covariogram realisability problems

For a deterministic set A, one calls *local covariogram* of A the map

$$\begin{array}{rcl} \mathbb{R}^d \times \mathscr{W} & \to & \mathbb{R} \\ (y;W) & \mapsto & \delta_{y;W}(A) := \mathcal{L}(A \cap (y+A) \cap W) \end{array} \tag{9.5}$$

where $\mathscr{W}$ denotes the set of *observation windows* defined by

$$\mathscr{W} = \left\{ W \subset \mathbb{R}^d \text{ bounded open set such that } \mathcal{L}(\partial W) = 0 \right\}.$$

Given a RAMS X, we denote by $\gamma_{\mathsf{X}}(y;W) = \mathbf{E}\delta_{y;W}(\mathsf{X})$ the *(mean) local covariogram* of X. If X is stationary, then the map $W \mapsto \gamma_{\mathsf{X}}(y;W)$ is translation invariant and extends into a measure proportional to the Lebesgue measure. Hence, one calls *specific covariogram* of X and denotes by $y \mapsto \gamma^s_{\mathsf{X}}(y)$, the map such that $\gamma_{\mathsf{X}}(y;W) = \mathbf{E}\delta_{y;W}(\mathsf{X}) = \gamma^s_{\mathsf{X}}(y)\mathcal{L}(W)$. Note that one simply has $\gamma^s_{\mathsf{X}}(y) = \gamma_{\mathsf{X}}(y,(0,1)^d)$.

We are interested in this chapter in the specific covariogram realisability problem: Given a function $S_2 : \mathbb{R}^d \to \mathbb{R}$, does there exists a stationary random measurable set $\mathsf{X} \in \mathscr{B}_d$ such that $S_2(y) = \gamma^s_{\mathsf{X}}(y)$ for all $y \in \mathbb{R}^d$?

The specific covariogram candidate S_2 has to verify some structural necessary condition to be realisable.

Definition 86 (Covariogram admissible functions)**.** A function $\gamma : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}$ is said to be *$\mathscr{B}_d$-local covariogram admissible*, or just *admissible*, if for all 5-tuples $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathscr{W}^q, c \in \mathbb{R})$,

$$\left[\forall A \in \mathscr{B}_d, \quad c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(A) \geq 0 \right] \Rightarrow c + \sum_{i=1}^{q} a_i \gamma(y_i;W_i) \geq 0.$$

A function $S_2 : \mathbb{R}^d \to \mathbb{R}$ is said to be *$\mathscr{B}_d$-specific covariogram admissible*, or just *admissible*, if the function $(y;W) \mapsto S_2(y)\mathcal{L}(W)$ is *$\mathscr{B}_d$-local covariogram admissible.*

It is an immediate consequence of the positivity and linearity of the mathematical expectation that a realisable S_2 function is necessarily admissible. Checking whether a given S_2 is admissible, a problem of combinatorial nature, is difficult. It will not be addressed here, but as emphasized in (9.3), it is directly related to the positivity problem for two-point covering functions, which is studied in numerous works (see [DL97; She63; Mat93; Qui08],[Lac15], and references therein). Let us remark that being admissible is a strong constraint on S_2 that conveys the usual properties of covariogram functions, and in particular, $S_2(y) \geq 0$ for all $y \in \mathbb{R}^d$ (since for all $y \in \mathbb{R}^d$, $W \in \mathscr{W}$ and $A \in \mathscr{B}_d$, $\delta_{y;W}(A) \geq 0$).

In general, the admissibility of S_2 is not sufficient for S_2 to be realisable. Consider the linear operator Φ

$$\Phi\left(c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i} \right) = c + \sum_{i=1}^{q} a_i S_2(y_i)\mathcal{L}(W_i) \tag{9.6}$$

on the subspace of functionals on $\mathscr{B}_d$ generated by the constant functions and the covariogram evaluations $A \mapsto \delta_{y;W}(A)$, $y \in \mathbb{R}^d$, $W \in \mathscr{W}$. The realisability of S_2 corresponds to the existence of a probability measure μ on $\mathscr{B}_d$ representing Φ, i.e. such that $\Phi(g) = \int_{\mathscr{B}_d} g d\mu$ for g in the aforementioned subspace. In a non-compact space such as $\mathscr{B}_d$, the positivity of Φ, i.e. the admissibility of S_2, is not sufficient to represent it by a probability measure, as the σ-additivity is also needed. See the previous chapter for a discussion and study of the realisability problem in an abstract framework.

It has been shown in [LM15] that in such non-compact frameworks, the realisability problem should better be accompanied with an additional condition involved with the regularity of the set in some sense. This condition is carried on by a companion function, called a *regularity modulus*, depending on the functions of interest in our realisability problem. Without entering into details (see Section 9.3), the perimeter function fulfils this role here, mostly because it can be approximated by linear combinations of covariograms, and has compact level sets within the space of measurable subsets of $\mathbb{R}^d$. The well-posed realisability problem with regularity condition we consider here deals with the existence of a stationary random measurable set $\mathsf{X} \in \mathscr{B}_d$ such that

$$\begin{cases} S_2(y) = \gamma_{\mathsf{X}}^s(y), \quad y \in \mathbb{R}^d, \\ \overline{\mathrm{Per}}(\mathsf{X}) = \mathbf{E}\mathrm{Per}(\mathsf{X}; (0,1)^d) < \infty. \end{cases}$$

The main result of this chapter is the following.

Theorem 87. *Let $S_2 : \mathbb{R}^d \mapsto \mathbb{R}$ be a function. Then S_2 is the specific covariogram of a stationary random measurable set $\mathsf{X} \in \mathscr{B}_d$ such that $\overline{\mathrm{Per}}(\mathsf{X}) < \infty$ if and only if S_2 is admissible and Lipschitz at* 0 *along the d canonical directions.*

This result is analogous to the one obtained in [LM15] for point processes, since the realisability condition is shown to be a positivity condition plus a regularity condition, namely the Lipschitz property of S_2. As already discussed, a realisable function S_2 is necessarily admissible. Besides, extending results from [Gal11], we show that a stationary RAMS X has a finite specific perimeter if and only if its specific covariogram γ_{X}^s is Lipschitz, and we obtain an explicit relation between the Lipschitz constant of S_2 and the specific perimeter. Hence the direct implication of Theorem 87 is somewhat straightforward. The real difficulty consists in proving the converse implication. To do so we adapt the techniques of [LM15] to our context which involves several technicalities regarding the approximation of the perimeter by linear combination of local covariogram functional. We first establish the counterpart of Theorem 87 for the realisability of local covariogram function $\gamma : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}$ (see Theorem 98) and we then extend this result to the case of specific covariogram of stationary RAMS (see Theorem 101).

In addition, we study the links between RAMS and the more usual framework of random closed sets (RACS), which *in fine* enables us to obtain a result analogous to Theorem 87 for RACS of the real line (see Theorem 104), such a result was out of reach with previously developped methods.

9.2 Random measurable sets

9.2.1 Definition of random measurable sets

Random measurable sets (RAMS) are defined as random variables taking value in the set $\mathscr{B}_d$ of Lebesgue (classes of) sets of $\mathbb{R}^d$ endowed with the Borel σ-algebra $\mathscr{B}(\mathscr{B}_d)$ induced by the natural topology, the so-called local convergence in measure. We recall that a sequence of measurable sets $(A_n)_{n\in\mathbf{N}}$ locally converges in measure to a measurable set A if for all bounded open sets $U \subset \mathbb{R}^d$, the sequence $\mathcal{L}\left((A_n \Delta A) \cap U\right)$ tends to 0, where Δ denotes the symmetric difference. The local convergence in measure simply corresponds to the convergence of the indicator functions $\mathbf{1}_{A_n}$ towards $\mathbf{1}_A$ in the space of locally integrable functions $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, and consequently $\mathscr{B}_d$ is a complete metrizable space [1].

[1] This is a consequence of the facts that $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a complete metrizable space and that the set of indicator functions is closed in $\mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Definition 88 (Random measurable sets)**.** A *random measurable set* (RAMS) X is a measurable map $\mathsf{X} : \omega \mapsto \mathsf{X}(\omega)$ from $(\Omega, \mathscr{O})$ to $(\mathscr{B}_d, \mathscr{B}(\mathscr{B}_d))$, where $\mathscr{B}(\mathscr{B}_d)$ denotes the Borel σ-algebra induced by the local convergence in measure.

Note that if X is a RAMS, then $\omega \mapsto \mathbf{1}_{\mathsf{X}(\omega)}$ is a random integrable function. This concept of random measurable (class of) set(s) is not standard. As mentioned in [Mol05], measurable random subsets of the interval $[0, 1]$ are defined following this definition in [SS87].

In the remaining part of this section, we will discuss the link between RAMS and other classical random objects, namely random Radon measures, measurable subsets of $\Omega \times \mathbb{R}^d$, and random closed sets.

Random Radon measures associated with random measurable sets Following the usual construction of random objects, a random Radon measure is defined as a measurable function from a probability space $(\Omega, \mathscr{O}, \mathbf{P})$ to the space $\mathbf{M}^+$ of positive Radon measures on $\mathbb{R}^d$ equipped with the smallest σ-algebra for which the evaluation maps $\mu \mapsto \mu(B)$, $B \in \mathscr{B}_d$ relatively compact, are measurable (see *e.g* [DV88b; Kal86; SW08]). Any RAMS $\mathsf{X} \subset \mathbb{R}^d$ canonically defines a random Radon measure that is the restriction to X of the Lebesgue measure, that is, $B \mapsto \mathcal{L}(\mathsf{X} \cap B)$ for Borel set $B \in \mathscr{B}_d$. The measurability of this restriction results from the observation that, for all $B \in \mathscr{B}_d$, the map $f \mapsto \int_B f(x)dx$ is measurable for the $\mathscr{L}^1_{\text{loc}}$-topology.

Existence of a measurable graph representative For a RAMS $\mathsf{X} : \Omega \to \mathscr{B}_d$, one can study the measurability properties of the graph $\mathsf{Y} = \{(\omega, x) : x \in \mathsf{X}(\omega)\} \subset \Omega \times \mathbb{R}^d$.

Definition 89 (Measurable graph representatives)**.** A subset $\mathsf{Y} \subset \Omega \times \mathbb{R}^d$ is a *measurable graph representative* of a RAMS X if

1. Y is a measurable subset of $\Omega \times \mathbb{R}^d$ (*i.e.* Y belongs to the product σ-algebra $\mathscr{O} \otimes \mathscr{B}_d$),
2. For a.a. $\omega \in \Omega$, the ω-section $\mathsf{Y}(\omega) = \{x \in \mathbb{R}^d,\ (\omega, x) \in \mathsf{Y}\}$ is equivalent in measure to $\mathsf{X}(\omega)$ (*i.e.* $\mathcal{L}(\mathsf{Y}(\omega) \Delta \mathsf{X}(\omega)) = 0$).

Proposition 90. *Any measurable set* $\mathsf{Y} \in \mathscr{O} \otimes \mathscr{B}_d$ *canonically defines a RAMS by considering the Lebesgue class of its* ω*-sections:*

$$\omega \mapsto \mathsf{Y}(\omega) = \{x \in \mathbb{R}^d,\ (\omega, x) \in \mathsf{Y}\}.$$

Conversely, any RAMS X *admits measurable graph representatives* $\mathsf{Y} \in \mathscr{O} \otimes \mathscr{B}_d$.

Proof. The first point is trivial. Let us prove the second point. Consider the random Radon measure μ associated to X, that is

$$\mu(\omega, B) = \mathcal{L}(\mathsf{X}(\omega) \cap B) = \int_B \mathbf{1}_{\mathsf{X}(\omega)}(x)dx.$$

By construction this random Radon measure is absolutely continuous with respect to the Lebesgue measure. Then according to Radon-Nikodym theorem for random measures (see the Appendix in [GL15]), there exists a jointly measurable map $g : (\Omega \times \mathbb{R}^d, \mathscr{O} \otimes \mathscr{B}_d) \to \mathbb{R}$ such that for all $\omega \in \Omega$,

$$\mu(\omega, B) = \int_B g(\omega, x)dx, \quad B \in \mathscr{B}_d.$$

Hence for all $\omega \in \Omega$, $\mathbf{1}_{\mathsf{X}(\omega)}(\cdot)$ and $g(\omega, \cdot)$ are both Radon-Nikodym derivative of $\mu(\omega, \cdot)$ and thus are equal almost everywhere. In particular, for a.a. $x \in \mathbb{R}^d$, $g(\omega, x) \in \{0, 1\}$.

Consequently, the function $(\omega, x) \mapsto \mathbf{1}(g(\omega, x) = 1)$ is also jointly measurable and is a Radon-Nikodym derivative of $\mu(\omega, \cdot)$ for all $\omega \in \Omega$, and thus the set

$$\mathsf{Y} = \{(\omega, x) \in \Omega \times \mathbb{R}^d,\ g(\omega, x) = 1\}$$

is a measurable graph representative of X. $\square$

Remark 91. Given a measurable graph representative Y of a RAMS X, one can consider measurable events of Ω related to Y. However, only a subset of these events belongs to the sub-σ-algebra $\sigma(\mathsf{X})$ of $\mathscr{B}(\mathscr{B}_d)$ induced by X. For example, for a given $x \in \mathbb{R}^d$, Fubini theorem ensures that the x-section set $\{\omega \in \Omega,\ (\omega, x) \in \mathsf{Y}\}$ is a measurable subset of Ω, but it depends on the Lebesgue representative chosen for X (e.g. $\mathsf{Y} \setminus (\Omega \times \{x\})$ is another measurable representative for which the x-section set is always empty). However, events such as $\{\omega \in \Omega,\ \mathcal{L}(\mathsf{Y}(\omega, \cdot) \cap B) \geq a\}$, for some $B \in \mathscr{B}_d$ and $a > 0$, are events of $\sigma(\mathsf{X})$ since they are invariant by a change of Lebesgue representative.

Random measurable sets and random closed sets Recall that $(\Omega, \mathscr{O}, \mathbf{P})$ denotes our probability space. Let $\mathscr{F} = \mathscr{F}\left(\mathbb{R}^d\right)$ be the set of all closed subsets of $\mathbb{R}^d$. Following [Mol05, Definition 1.1] a random closed set is defined as follows.

Definition 92 (Random closed sets)**.** A map $\mathsf{Z} : \Omega \to \mathscr{F}$ is called a *random closed set* (RACS) if for every compact set $K \subset \mathbb{R}^d$, $\{\omega :\ \mathsf{Z}(\omega) \cap K \neq \emptyset\} \in \mathscr{O}$.

The framework of random closed sets is standard in stochastic geometry [Mat75; Mol05]. Let us now reproduce a result of C.J. Himmelberg that allows to link the different notions of random sets (see [Mol05, Theorem 2.3] or the original chapter [Him75] for the complete theorem).

Theorem 93 (Himmelberg)**.** *Let $(\Omega, \mathscr{O}, \mathbf{P})$ be a probability space and $\mathsf{Z} : \Omega \to \mathsf{Z}(\omega) \in \mathscr{F}$ be a map taking values into the set of closed subsets of $\mathbb{R}^d$. Consider the two following assertions:*

(i) $\{\omega :\ \mathsf{Z} \cap F \neq \emptyset\} \in \mathscr{O}$ *for every closed set* $F \subset \mathbb{R}^d$*,*

(ii) The graph of Z, i.e. the set $\{(\omega, x) \in \Omega \times \mathbb{R}^d :\ x \in \mathsf{Z}(\omega)\}$, belongs to the product σ-algebra $\mathscr{O} \otimes \mathscr{B}_d$,

Then the implication $(i) \Rightarrow (ii)$ is always true, and if the probability space $(\Omega, \mathscr{O}, \mathbf{P})$ is complete (i.e. all $\mathbf{P}$-negligible subsets of Ω are measurable) one has the equivalence $(i) \Leftrightarrow (ii)$.

In view of our definitions for random sets, Himmelberg's theorem can be rephrased in the following terms.

Proposition 94 (RACS and closed RAMS)**.** *(i) Any RACS Z has a measurable graph $\mathsf{Y} = \{(\omega, x) \in \Omega \times \mathbb{R}^d :\ x \in \mathsf{Z}(\omega)\}$, and thus also defines a unique random measurable set.*

(ii) Suppose that the probability space $(\Omega, \mathscr{O}, \mathbf{P})$ is complete. Let $\mathsf{Y} \in \mathscr{O} \otimes \mathscr{B}_d$ be a measurable set such that for all $\omega \in \Omega$, its ω-section $\mathsf{Y}(\omega) = \{x \in \mathbb{R}^d,\ (\omega, x) \in \mathsf{Y}\}$ is a closed subset of $\mathbb{R}^d$. Then, the map $\omega \mapsto \mathsf{Y}(\omega)$ defines a random closed set.

9.2.2 Random measurable sets of finite perimeter

For a closed set F, the perimeter is generally defined by the $(d-1)$-dimensional measure of the topological boundary, that is $\mathcal{H}^{d-1}(\partial F)$. This definition is not relevant for a measurable set $A \subset \mathbb{R}^d$, in the sense that the value $\mathcal{H}^{d-1}(\partial A)$ strongly depends on the representative of A within its Lebesgue class. The proper notion of perimeter for measurable sets is the variational perimeter that defines the perimeter as the variation of the indicator function of the set. An important feature of the variational perimeter is that it is lower semi-continuous for the convergence in measure, while the functional $F \mapsto \mathcal{H}^{d-1}(\partial F)$ is not lower semi-continuous on the set of closed sets $\mathscr{F}$ endowed with the hit or miss topology. This is a key aspect for this chapter since it allows to consider the variational perimeter as a regularity modulus for realisability problems in following the framework of [LM15].

Variational perimeters Let U be an open subset of $\mathbb{R}^d$. Recall that the *(variational) perimeter* $\operatorname{Per}(A;U)$ of a measurable set $A \in \mathscr{B}_d$ in the open set U is defined by (9.4). Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$. Closely related to the perimeter, one also defines the *directional variation* in the direction $u \in \mathbb{S}^{d-1}$ of A in U by [AFP00, Section 3.11]

$$V_u(A;U) = \sup\left\{\int_U \mathbf{1}_A(x)\langle\nabla\varphi(x),u\rangle dx:\ \varphi\in\mathscr{C}_c^1(U,\mathbb{R}),\ |\varphi(x)|\leq 1 \text{ for all } x\right\}.$$

For technical reasons, we also consider the *anisotropic perimeter*

$$A \mapsto \operatorname{Per}_{\mathbf{B}}(A;U) = \sum_{j=1}^{d} V_{e_j}(A;U)$$

which adds up the directional variations along the d directions of the canonical basis $\mathbf{B} = \{e_1,\dots,e_d\}$. In geometric measure theory, the functional $A \mapsto \operatorname{Per}_{\mathbf{B}}(A;U)$ is described as the anisotropic perimeter associated with the anisotropy function $x \mapsto \|x\|_\infty$ (see e.g. [Cas+08] and the references therein). Indeed, one easily sees that

$$\operatorname{Per}_{\mathbf{B}}(A;U) = \sup\left\{\int_U \mathbf{1}_A(x)\operatorname{div}\varphi(x)dx:\ \varphi\in\mathscr{C}_c^1(U,\mathbb{R}^d),\ \|\varphi(x)\|_\infty\leq 1 \text{ for all } x\right\}.$$

Hence the only difference between the variational definition of the isotropic perimeter $\operatorname{Per}(A;U)$ and the one of the anisotropic perimeter $\operatorname{Per}_{\mathbf{B}}(A;U)$ is that the test functions φ take values in the ℓ_2-unit ball B_d for the former whereas they take values in the ℓ_∞-unit ball $[-1,1]^d$ for the latter. The set inclusions $B_d \subset [-1,1]^d \subset \sqrt{d}B_d$ lead to the tight inequalities

$$\operatorname{Per}(A;U) \leq \operatorname{Per}_{\mathbf{B}}(A;U) \leq \sqrt{d}\operatorname{Per}(A;U). \tag{9.7}$$

Consequently a set A has a finite perimeter $\operatorname{Per}(A;U)$ in U if and only if it has a finite anisotropic perimeter $\operatorname{Per}_{\mathbf{B}}(A;U)$ (let us mention that this equivalence is not true when considering only one directional variation $V_u(A;U)$). One says that a measurable set $A \subset \mathbb{R}^d$ has *locally finite perimeter* if A has a finite perimeter $\operatorname{Per}(A;U)$ in all bounded open sets $U \subset \mathbb{R}^d$.

To finish let us mention that if X is a RAMS then $\operatorname{Per}(\mathsf{X};U)$, $\operatorname{Per}_{\mathbf{B}}(\mathsf{X};U)$, and $V_u(\mathsf{X};U)$, $u \in \mathbb{S}^{d-1}$, are well-defined random variables since the maps $A \mapsto \operatorname{Per}(A;U)$, $A \mapsto \operatorname{Per}_{\mathbf{B}}(A;U)$ and $A \mapsto V_u(A;U)$ are lower semi-continuous for the convergence in measure [AFP00]. Consequently one says that a RAMS X has a.s. finite (resp. locally finite) perimeter in U if the random variable $\operatorname{Per}(\mathsf{X};U)$ is a.s. finite (resp. if for all bounded open sets $V \subset U$ $\operatorname{Per}(\mathsf{X};V)$ is a.s. finite).

Remark 95. Rataj recently proposed a framework for "random sets of finite perimeter" [Rat14] that models random sets as random variables in the space of indicator functions of sets of finite perimeter endowed with the Borel σ-algebra induced by the strict convergence in the space of functions of bounded variation [AFP00, Section 3.1]. Since this convergence induced the $\mathscr{L}^1$-convergence of indicator functions, any "random set of finite perimeter" X uniquely defines a RAMS X having a.s. finite perimeter. One advantage of the RAMS framework is that it is more general in the sense that it enables to consider random sets that do not have finite perimeter.

Closed representative of one-dimensional sets of finite perimeter Although the general geometric structure of sets of finite perimeter is well-known (see [AFP00, Section 3.5]) it necessitates involved notions from geometric measure theory (rectifiable sets, reduced and essential boundaries, etc.). However, when restricting to the case of one-dimensional sets of finite perimeter, all the complexity vanishes since subsets of $\mathbb{R}$ having finite perimeter all correspond to finite unions of non empty and disjoint closed intervals.

More precisely, according to Proposition 3.52 of [AFP00], if a non-negligible measurable set $A \subset \mathbb{R}$ has finite perimeter in an interval $(a,b) \subset \overline{\mathbb{R}}$, there exists an integer p and p pairwise disjoint non empty and closed intervals $J_i = [a_{2i-1}, a_{2i}] \subset \overline{\mathbb{R}}$, with $a_1 < a_2 < \cdots < a_{2p}$ such that

- $A \cap (a,b)$ is equivalent in measure to the union $\bigcup_i J_i$,
- the perimeter of A in (a,b) is the number of interval endpoints belonging to (a,b)

$$\mathrm{Per}(A;(a,b)) = \#\{a_1, a_2, \dots, a_{2p}\} \cap (a,b).$$

Let us remark that a set of the form $A = \bigcup_i [a_{2i-1}, a_{2i}]$ is closed and that such a set satisfies the identity $\mathrm{Per}(A;(a,b)) = \mathcal{H}^0(\partial A \cap (a,b))$, where ∂A denotes the topological boundary of A and $\mathcal{H}^0$ is the Hausdorff measure of dimension 0 on $\mathbb{R}$ (*i.e.* the counting measure) while in the general case one only has $\mathrm{Per}(A;(a,b)) \leq \mathcal{H}^0(\partial A \cap (a,b))$ since A may contain isolated points.

More generally, if $A \subset \mathbb{R}$ has locally finite perimeter, then there exists a unique countable or finite family of closed and disjoint intervals $J_i = [a_{2i-1}, a_{2i}]$, $i \in I \subset \mathbb{Z}$, such that A is equivalent in measure to $\bigcup_{i\in I} J_i$ and for all bounded open intervals (a,b), $\mathrm{Per}(A;(a,b))$ is the number of interval endpoints belonging to (a,b).

Using both this observations and Proposition 94, one obtains the following proposition.

Proposition 96. *Suppose that the probability space $(\Omega, \mathscr{O}, \mathbf{P})$ is complete. Let X be a RAMS of $\mathbb{R}$ that has a.s. locally finite perimeter. Then, there exists a RACS $\mathsf{Z} \subset \mathbb{R}$ such that for $\mathbf{P}$-almost all $\omega \in \Omega$ and for all $a < b \in \mathbb{R}$,*

$$\mathcal{L}^1(\mathsf{X}(\omega)\Delta\mathsf{Z}(\omega)) = 0 \quad \textit{and} \quad \mathrm{Per}(\mathsf{X}(\omega);(a,b)) = \mathcal{H}^0(\partial\mathsf{Z}(\omega) \cap (a,b)).$$

Non-closed RAMS in dimension $d > 1$ In contrast to the one-dimensional case, in dimension $d > 1$ there exist measurable sets of finite perimeter that do not have closed representative in their Lebesgue class. In [AFP00, Example 3.53], Ambrosio *et al.* consider a set obtained as the union of an infinite family of open balls with small radii and with centers forming a dense subset of $[0,1]^d$, which yields the following result.

Proposition 97. *There exists a measurable subset A of $[0,1]^d$ with finite perimeter, finite measure $\mathcal{L}(A) < 1$, and such that $\mathcal{L}(A \cap U) > 0$ for any open subset U of $[0,1]^d$.*

Such a set clearly has no closed representative, because if it had one, say F, then F would charge every open subset of $[0,1]^d$, and therefore it would be dense in $[0,1]^d$. But since F is closed, one would have $F = [0,1]^d$, which contradicts $\mathcal{L}(F) = \mathcal{L}(A) < 1$.

9.3 Realisability result

9.3.1 Realisability problem and regularity modulus

Recall that the local covariogram of a RAMS X is $\gamma_{\mathsf{X}}(y;W) = \mathbf{E}\delta_{y;W}(\mathsf{X})$. Let us introduce a regularized realisability problem for local covariogram. Put $U_n = (-n,n)^d$. Define the *weighted anisotropic perimeter* by

$$\mathrm{Per}_{\mathbf{B}}^{\beta}(A) = \sum_{n\geq 1} \beta_n \mathrm{Per}_{\mathbf{B}}(A;U_n)$$

where the sequence (β_n) is set to $\beta_n = 2^{-n}(2n)^{-d}$ so that $\sum_{n\geq 1}\beta_n \mathcal{L}(U_n) = 1$. For a given function $\gamma : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}$, define

$$\sigma_\gamma(u;W) = \frac{1}{\|u\|}[\gamma(0;W\ominus[-u,0]) - \gamma(u;W\ominus[-u,0]) + \gamma(0;W\ominus[0,u]) - \gamma(-u;W\ominus[0,u])].$$

One defines for all windows $W\in\mathscr{W}$ the constant $L_j(\gamma,W)\in[0,+\infty]$ by

$$L_j(\gamma,W) = \sup_{\varepsilon\in\mathbb{R}} \sigma_\gamma(\varepsilon e_j;W), \quad j\in\{1,\dots,d\}. \tag{9.8}$$

$L_j(\gamma,W)$ is related to the Lipschitz property of γ in its spatial variable. The motivation for considering this particular constant is that if γ_{X} is the local covariogram of a RAMS X, then

$$\mathbf{E}V_{e_j}(\mathsf{X};W) = \sup_{\varepsilon\in\mathbb{R}} \sigma_{\gamma_{\mathsf{X}}}(\varepsilon e_j;W).$$

Theorem 98. *Let $\gamma : \mathbb{R}^d\times\mathscr{W}\to\mathbb{R}$ be a function and $r\geq 0$. Then γ is realisable by a RAMS X such that*

$$\mathbf{E}\mathrm{Per}_{\mathbf{B}}^{\beta}(\mathsf{X}) \leq r$$

if and only if γ is admissible (see Definition 86) and

$$\sum_{n\geq 1}\beta_n\left(\sum_{j=1}^{d} L_j(\gamma,U_n)\right) \leq r, \tag{9.9}$$

where for all $j\in\{1,\dots,d\}$ and $n\geq 1$, the constant $L_j(\gamma,U_n)$ is defined by (9.8)*.*

The stationary counterpart of the above theorem is stated and proved in Section 9.3.2.

Let us recall the general definition and result of [LM15] that we use to prove Theorem 98.

Definition 99 (Regularity moduli)**.** Let $\mathscr{G}$ be a vector space of measurable real functions on $\mathscr{B}_d$. A *$\mathscr{G}$-regularity modulus* on $\mathscr{B}_d$ is a lower semi-continuous function $\chi : \mathscr{B}_d \mapsto [0,+\infty]$ such that for all $g\in\mathscr{G}$, the level set

$$H_g = \{A\in\mathscr{B}_d,\ \chi(A)\leq g(A)\} \subset \mathscr{B}_d$$

is relatively compact for the convergence in measure.

In our setting, call $\mathscr{G}$ the vector space generated by the constant functionals and the local covariogram functionals $A\mapsto\delta_{y;W}(A)$, $y\in\mathbb{R}^d$, $W\in\mathscr{W}$.

Proposition 100. $\mathrm{Per}_{\mathbf{B}}^{\beta}$ *is a $\mathscr{G}$-regularity modulus (and therefore a $\mathscr{G}^*$-regularity modulus for any subspace $\mathscr{G}^*\subset\mathscr{G}$).*

Proof. By definition of a regularity modulus, one has to show that the $\mathrm{Per}^{\beta}_{\mathbf{B}}$-level sets are relatively compact. Consider a sequence (A_n) such that $\mathrm{Per}^{\beta}_{\mathbf{B}}(A_n) \leq c$ for all $n \in \mathbf{N}$. Then for all $n, m \in \mathbf{N}$, $\mathrm{Per}_{\mathbf{B}}(A_n; U_m) \leq \frac{c}{\beta_m} < \infty$ and thus (A_n) is a sequence of sets of locally finite perimeter whose perimeter in any open bounded set $U \subset \mathbb{R}^d$ is uniformly bounded. According to Theorem 3.39 of [AFP00], there exists a subsequence of (A_n) that locally converges in measure in $\mathbb{R}^d$. $\square$

For $g \in \mathscr{G}$, denote by $\mathrm{dom}(g)$ the smallest open set such that for every measurable set A, $g(A) = g(A \cap \mathrm{dom}(g))$. If g has the form

$$g = \sum_{i=1}^{q} a_i \delta_{y_i; W_i}, \tag{9.10}$$

we have $\mathrm{dom}(g) \subset \cup_i (W_i \cup (-y_i + W_i))$, but there is not equality because such a decomposition is not unique.

9.3.2 Stationary case

The following theorem is the main result of this chapter. It is a refined version of Theorem 87 given in the introduction.

Theorem 101. *Let $S_2 : \mathbb{R}^d \to \mathbb{R}$ be a function and $r \geq 0$. Then there is a stationary RAMS X such that*

$$\begin{cases} S_2(y) = \gamma^s_{\mathsf{X}}(y), & y \in \mathbb{R}^d, \\ \mathrm{Per}^s_{\mathbf{B}}(\mathsf{X}) \leq r \end{cases} \tag{9.11}$$

if and only if S_2 is admissible and

$$\sum_{j=1}^{d} \mathrm{Lip}_j(S_2, 0) \leq \frac{r}{2}.$$

We shall use a variant of Theorem 2.10(ii) from [LM15], where the monotonicity assumption is replaced by a domination.

Theorem 102. *Let $\mathscr{G}^*$, χ, Φ be like in Theorem 75 and assume that $\mathscr{G}^*$ is stable under the action of a group of transformations Θ of $\mathbb{R}^d$: For all $\theta \in \Theta$, $g \in \mathscr{G}^*$, $\theta g : A \mapsto g(\theta A)$ is a function of $\mathscr{G}^*$. Assume furthermore that there is a sequence $(g_n)_{n \geq 1}$ of functions of $\mathscr{G}^*$ such that $0 \leq g_n \leq \chi$ and*

$$g_n(A) \underset{n \to +\infty}{\longrightarrow} \chi(A), \quad A \in \mathscr{B}_d,$$

and that χ is sub-invariant*: For every $\theta \in \Theta$, there is a constant $C_\theta > 0$ such that*

$$\chi(\theta A) \leq C_\theta \chi(A), \quad A \in \mathbb{M}. \tag{9.12}$$

Then if Φ is invariant under the action of Θ, that is,

$$\Phi(\theta g) = \Phi(g), \quad g \in \mathscr{G}^*, \ \theta \in \Theta,$$

for any given $r \geq 0$, there exists a Θ-invariant RAMS X such that

$$\begin{cases} \mathbf{E} g(\mathsf{X}) = \Phi(g), & g \in \mathscr{G}^*, \\ \mathbf{E} \chi(\mathsf{X}) \leq r \end{cases}$$

if and only if (8.5) *holds.*

9.3.3 Covariogram realisability problem for RACS of $\mathbb{R}$

The goal of this section is to establish a result similar to Theorem 101 for the specific covariogram of one-dimensional stationary RACS.

First let us discuss the definition of local covariogram admissibility of functions in arbitrary dimension $d \geq 1$. By analogy with the definition of $\mathscr{B}_d$-local covariogram admissible functions (see Definition 86), when considering RACS of $\mathbb{R}^d$, that is random variables taking values in $\mathscr{F} = \mathscr{F}(\mathbb{R}^d)$ the set of all closed subsets of $\mathbb{R}^d$, one says that a function $\gamma : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}$ is *$\mathscr{F}$-local covariogram admissible* if for all 5-tuples $(q \geq 1, (a_i) \in \mathbb{R}^q, (y_i) \in (\mathbb{R}^d)^q, (W_i) \in \mathscr{W}^q, c \in \mathbb{R})$,

$$\left[\forall F \in \mathscr{F}, \quad c + \sum_{i=1}^{q} a_i \delta_{y_i;W_i}(F) \geq 0\right] \Rightarrow c + \sum_{i=1}^{q} a_i \gamma(y_i; W_i) \geq 0.$$

Besides, one says that $S_2 : \mathbb{R} \to \mathbb{R}$ is *$\mathscr{F}$-specific covariogram admissible* if $(y, W) \mapsto S_2(y)\mathcal{L}(W)$ is $\mathscr{F}$-local covariogram admissible. However this distinction is superfluous since these two notions of admissibility are strictly equivalent.

Proposition 103. *A function $\gamma : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}$ is $\mathscr{F}$-local covariogram admissible if and only if it is $\mathscr{B}_d$-local covariogram admissible.*

Now that this technical point has been clarified we are in position to formulate our result for the realisability of specific covariogram of stationary RACS of $\mathbb{R}$.

Theorem 104. *Suppose that the probability space $(\Omega, \mathscr{O}, \mathbf{P})$ is complete. Let $S_2 : \mathbb{R} \to \mathbb{R}$ be a given function and let $r > 0$. Then S_2 is the covariogram of a stationary RACS $Z \subset \mathbb{R}$ such that*

$$\mathbf{E}\left(\mathcal{H}^0(\partial Z) \cap (0, 1)\right) \leq r$$

if and only if S_2 is $\mathscr{F}$-specific covariogram admissible and Lipschitz with Lipschitz constant $L \leq \frac{r}{2}$.

Note that although the geometry of sets with finite perimeter on the line seems quite simplistic, a direct proof of the realisability result above is far from trivial.

Chapter 10

Inhomogeneous intensity estimation with Voronoi tessellations [Mor+19]

Abstract: Voronoi estimators are non-parametric and adaptive estimators of the intensity of a point process. The intensity estimate at a given location is equal to the reciprocal of the size of the Voronoi/Dirichlet cell containing that location. Their major drawback is that they tend to paradoxically under-smooth the data in regions where the point density of the observed point pattern is high, and over-smooth where the point density is low. To remedy this behaviour, we propose to apply an additional smoothing operation to the Voronoi estimator, based on resampling the point pattern by independent random thinning. Through a simulation study we show that our resample-smoothing technique improves the estimation substantially. In addition, we study statistical properties such as unbiasedness and variance, and propose a rule-of-thumb and a data-driven cross-validation approach to choose the amount of smoothing to apply. Finally we apply our proposed intensity estimation scheme to two datasets: locations of pine saplings (planar point pattern) and motor vehicle traffic accidents (linear network point pattern).

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