Replication of arithmetic random waves

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Helmholtz equation

Eigenmodes : Solutions F_k of

$$\Delta F + k^2 F = 0$$

- Δ : Laplacian operator on a manifold (here $\mathbb{R}^2$ or $\mathbb{T}^2$)
- k : wavenumber
- Spatial component of solutions of d'Alembert wave propagation equation
- on $\mathbb{R}$: $F_k(x) = a\cos(kx) + b\sin(kx)$
- on $\mathbb{R}^2$: for instance

$$F_k(x) = \sum_j a_j \cos(k\langle \underbrace{u_j}_{\text{unit norm}}, x\rangle) + b_j \sin(k\langle \underbrace{u_j}_{\text{unit norm}}, x\rangle)$$

$$F_k(x) = \int_{\mathbb{S}^1} [a_u \cos(k\langle x, u\rangle) + b_u \sin(k\langle x, u\rangle)]\, \mu(du) \text{ for } \mu \text{ finite measure}$$

Chadlni figures and nodal lines

Object of interest : nodal lines :

$$\mathcal{L}_k = F_k^{-1}(\{0\})$$

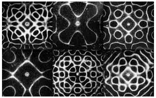

Figure – https://www.youtube.com/watch?v=tQ2fvciSf_0

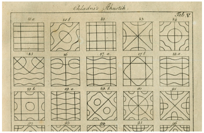

Figure – Chladni figures

Eigenmodes on $\mathbb{T}^2$

- Let $f_u(x) := \cos(2\pi\langle u, x\rangle)$ or $\sin(2\pi\langle u, x\rangle)$, $u \in \mathbb{R}^2$
- f_u continuous on $\mathbb{T}^2 \Leftrightarrow f$ is $(1,1)$-periodic $\Leftrightarrow u \in \mathbb{Z}^2$
- $\Delta f_u(x) = -4\pi^2(u_1^2 + u_2^2) f(x)$
- For $n \in \mathbb{N}$,

$$\mathcal{E}_n = \{f_u : \|u\|^2 = n\} \Rightarrow \text{ Solutions of } \Delta f + 4\pi^2 nf = 0$$

- $\mathscr{S} \subset \mathbb{N} := \{n : \mathcal{E}_n \neq 0\}$ with

$$\#\mathcal{E}_n = \begin{cases} 0 \text{ if some } q_i \text{ has odd valuation} \\ 4\prod_{i=1}^m (1+\alpha_i) \text{ otherwise} \end{cases}$$

where

$$n = 2^\alpha p_1^{\alpha_1} \dots p_m^{\alpha_m} q_1^{2\beta_1} \dots . q_l^{2\beta_l}$$

with $p_i \equiv 1[4]$, $q_i \equiv 3[4]$

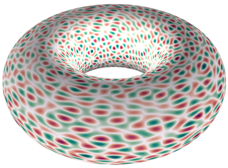

Figure – Arithmetic Random Wave (Credit : Simon Coste)

Typical eigenfunctions

- For a density 1 subsequence $\mathscr{S}'$ of integers $n \subset \mathscr{S}$,

$$\mathcal{N}_n = \ln(n)^{\ln(2)/2+o(1)}$$

- Let $F_n : \sqrt{n}\mathbb{T}^2 \to \mathbb{R}$ a random "Planck scale" Gaussian eigenfunction :

$$F_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{u:\|u\|^2=n} \underbrace{a_u}_{iid\ \mathcal{N}(0,1)} \cos\left(\left\langle x, \frac{u}{\sqrt{n}}\right\rangle\right) + \underbrace{b_u}_{iid\ \mathcal{N}(0,1)} \sin\left(\left\langle x, \frac{u}{\sqrt{n}}\right\rangle\right)$$

- The covariance function is

$$\begin{aligned} r_n(x-y) &= \mathrm{Cov}(F_n(x), F_n(y)) = \mathbb{E}(F_n(x)F_n(y)) \\ &= \frac{1}{\mathcal{N}_n} \sum_{u\in\mathbb{Z}^2:\|u\|^2=n} \cos(\langle x-y, \frac{u}{\sqrt{n}}\rangle) \end{aligned}$$

Convergence of the covariance function

$$r_n(x) = \frac{1}{\mathcal{N}_n} \sum_{u \in \mathbb{Z}^2 : \|u\|^2 = n} \cos(\langle x, u/\sqrt{n} \rangle) = \int_{\mathbb{S}^1} \cos(\langle x, u \rangle) d\mu_n(u)$$

$$\text{where } \mu_n := \frac{1}{\mathcal{N}_n} \sum_{u \in \mathbb{Z}^2 : \|u\|^2 = n} \delta_{u/\sqrt{n}} \Rightarrow \mu_{\mathbb{S}^1} \text{ Haar measure on } \mathbb{S}^1$$

for $n \in \mathscr{S}'' \subset \mathbb{N}$ of density 1 (not trivial !). Pointwise convergence to the $0-$Bessel function

$$r_n(x) \to J_0(x) = \int \cos(\langle x, u \rangle) d\mu_{\mathbb{S}^1}(u)$$

Remark : J_0 is the covariance function of an isotropic stationary field on $\mathbb{R}^2$, the **Random planar wave model** :

$$\text{Cov}(f^{RPW}(x), f^{RPW}(y)) = J_0(x - y)$$

Berry's conjecture on nodal lines

Expectation : **Oravecz, Rudnick and Wigman '08**

$$\mathcal{L}_B := \text{length}(F_n^{-1}(\{0\}) \cap B), B \subset \sqrt{n}\mathbb{T}^2$$

$$\mathbb{E}(\mathcal{L}_B) = |B| \frac{1}{2\sqrt{2}}$$

Berry's conjecture, proved by **Krishnapur, Kurlberg, Wigman** (Annals of Mathematics, 2011), for $n \in \mathscr{S}'$

$$\text{Var}(\mathcal{L}_{\sqrt{n}\mathbb{T}^2}) \sim \frac{c_n}{512} \frac{n^2}{\mathcal{N}_n^2} \text{ where } c_n \in [1/2, 1] \text{ “oscillates” as } n \to \infty$$

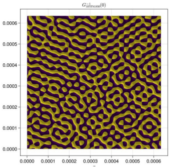

Figure – Nodal lines (L. Thomassey)

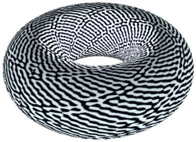

Figure – Excursion (Simon Coste)

Small balls and full correlation

- Generalisation by **Benatar, Marinucci, Wigman 2020** to small balls : For $\alpha > 0, s_n > n^\alpha$,

$$\mathcal{L}_{s_n} := \text{length}(F_n^{-1}(\{0\}) \cap B(s_n))$$
$$\text{Var}(\mathcal{L}_{s_n}) \sim c_k |B(s_n)|^2 \frac{1}{\mathcal{N}_n^2}$$

- Furthermore, there is full correlation between small balls and $\sqrt{n}\mathbb{T}^2$:

$$\sup_{s \geqslant n^\alpha} |\text{Corr}(\mathcal{L}_s, \mathcal{L}_{\sqrt{n}\mathbb{T}^2}) - 1| \to 0.$$

- Based on the Kac-Rice formula and tiedous computations of the "spectral quasi-correlations", i.e.

$$\#\{(u_1, \dots, u_l) \in (\mathbb{Z}^2)^l : 0 < |u_1 + \cdots + u_l| < \varepsilon, \|u_i\|^2 = n\}$$

Longstanding questions : Bourgain, Rudnick, Cileruello, Cordoba, ...

Phase transition

- There is "full correlation" at polynomial scales [BMW 20']
  Furthermore

$$\widetilde{\mathcal{L}}_{n^\alpha} := \frac{\mathcal{L}_{n^\alpha} - \mathbb{E}(...)}{\sqrt{\mathrm{Var}}} \to \text{ sum of Chi}^2 \text{ variables}$$

- Drastic change of behaviour below [**Dierickx, Nourdin, Peccati and Rossi '19**]

$$\widetilde{\mathcal{L}}_{\ln(n)^A} \to \mathcal{N}(0, \sigma^2) \text{ with } A = \frac{1}{18}\ln(\pi/2)$$

- There are conjectures about the phase transition, i.e. the minimal scale $\ln(n)^{A_c}$ where full correlation occurs :
  - [**Sartori '21**] Full correlation for $s_n = \ln(n)^B$ with $B = \frac{29}{6}\ln(2)$
  - Hence $A < A_c < B$
  - Sartori conjectures $A_c \geqslant \ln(\pi/2)/2$

What happens above the phase transition ?

Theorem (L,Thomassey 23+)

The covariance function is almost periodic at intermediates scales : there is an **almost period** τ_n *such that asymptotically*

$$\ln(n)^A \leqslant \tau_n \leqslant \exp(\mathcal{N}^{1+\varepsilon}) \; (<< n^\alpha)$$

and for $\beta \in \mathbb{N}^2$*, with high probability*

$$\sup_{t \in \sqrt{n}\mathbb{T}^2} |\partial^\beta F_n(t) - \partial^\beta F_n(t+\tau_n)| = o(\ln(n)^{-\delta}), \delta > 0$$

Remark : Much smaller than the actual period ($\sim \sqrt{n}$).

Almost periodicity

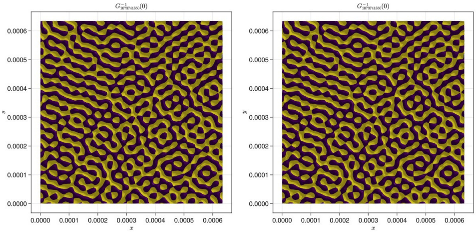

Figure – $n = 10^9$: Game of the 7 differences between F_n and $F_n^{\tau_n} = F_n(\tau_n + \cdot)$

Proof

1. Show that $r(\tau_n) > 1 - \exp(-\ln(n)^{\varepsilon})$
2. Use concentration results about suprema of random Gaussian fields

$$\sup_{x \in \sqrt{n}\mathbb{T}^2} \left| F_n - F_n^{\tau_n} \right|.$$

Consequences for nodal sets

- Geometric similarity : for φ with compact support,

$$\int_{F_n^{-1}(\{0\})} \varphi(t)\mathcal{H}^1(dt) - \int_{F_n^{-1}(\{0\})} \varphi(t+\tau_n)\mathcal{H}^1(dt) \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

To do list :

- Do we have with high probability

$$\text{Topology}(F_n^{-1}(\{0\}) \cap B) = \text{Topology}(F_n^{-1}(\{0\}) \cap (B+\tau_n))?$$

- Replication of *phase singularities*, i.e. (isolated) complex zeros of

$$F_n + \imath F_n'$$

where F_n' is an independent copy of F_n?

Dirichlet's principle

Lemma

Let $N > 1$ and

$$r(x) = \frac{1}{N} \sum_{i=1}^{N} \cos(2\pi \langle u_i, x \rangle) \text{ where } u_i \in \mathbb{R}^d, x \in \mathbb{R}^d.$$

Then for $a > 0$, for some $1 \leqslant \tau \leqslant a^{-N/d}$,

$$r(\tau) \geqslant 1 - ca^2$$

Application to ARW :

- $d = 2, N = \mathcal{N}_n = \ln(n)^{\ln(2)/2 + o(1)}$
- $u_i \in \mathbb{Z}^2$ such that $\|u_i\|^2 = n$,
- $a = \exp(-\ln(n)^{\varepsilon}) = \exp(-\mathcal{N}^{1+\varepsilon'}/\mathcal{N}) \Rightarrow \tau \leqslant \exp(\mathcal{N}^{1+\varepsilon'}/d)$

Dirichlet "Pigeon-hole" principle

- Strategy : Find some x such that all $\langle u_i, x\rangle$ are simultaneously close to an integer
- Try all $x \in [\![1, M]\!]^d$ for some M
- For each x, we have a vector

$$v_x := \Big(\langle x, u_i\rangle\ [\mathbb{Z}^d]\Big)_{i=1,\dots,\mathcal{N}} \in [0,1]^{\mathcal{N}}$$

- There are M^d possible x, hence if we divide $[0,1]^{\mathcal{N}}$ in $M^d + 1$ bins of diameter ε, say $v_x, v_{x'}$ end up in the same bin :

$$|v_x - v_{x'}| \leqslant \varepsilon$$

$$\text{dist}((\langle x, u_i\rangle)_i - (\langle x', u_i\rangle)_i, \mathbb{Z}^d) \leqslant \varepsilon \Rightarrow x - x' \text{ works !}$$

- Since there are about $\varepsilon^{-\mathcal{N}}$ bins of diameter ε,

$$\varepsilon^{-\mathcal{N}} = M^d + 1 \Rightarrow M \sim \varepsilon^{-\mathcal{N}/d}$$

Lower bound

- **[Dierickx, Nourdin, Peccati and Rossi '19]** : $r_n \to J_0$ uniformly on $B(\ln(n)^A)$, and

$$J_0(t) \xrightarrow[t \to 0]{} 0$$

we necessarily have $\tau_n > \ln(n)^A$. Can we do better ?

- Let $u_1, \ldots, u_{\mathcal{N}} \in \mathbb{S}^1$ random and

$$R_{\mathcal{N}}(x) = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \cos(\langle u_i, x \rangle).$$

- We want to show that for $\varepsilon < 1$, for $\tau_n \sim \exp(\mathcal{N})$

$$\sup_{\tau \in [0, \tau_n]} R_{\mathcal{N}}(\tau) < \varepsilon$$

it implies that pseudo-periods are at least of scale $\exp(\mathcal{N})$.

Lower bound

Theorem

Assumptions :

- *The system* $(u_1, \ldots, u_{\mathcal{N}})$ *is isotropic on* $\mathbb{S}^1$
- *The* $h(u_i)$ *satisfy the Hoeffding type inequality for* h *bounded smooth*

$$\mathbb{P}\left(\left|\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} h(u_i) - \mathbb{E}(h(u_1))\right| > t\right) < \exp(-ct^\gamma \mathcal{N})$$

where $\gamma, c > 0$ *do not depend on* h. *Then for* $\varepsilon > 0$

$$\sup_{\tau \in [\tau_0, \exp(\mathcal{N}^{1-\varepsilon})]} R_{\mathcal{N}}(\tau) < \frac{1}{2}.$$

Typical example : i.i.d. uniform u_i on $\mathbb{S}^1 (\gamma = 2)$.

Hence the proportion of $(u_1, \dots, u_{\mathcal{N}_n})$ such that $s_{\mathcal{N}_n}$ does not have a pseudo period $\tau_{\mathcal{N}_n} \sim \exp(\mathcal{N}_n^{1-\varepsilon})$ goes to 0.

- Either the $(u_1, \dots, u_{\mathcal{N}_n})$ such that $\|u_i\|^2 = n$ fall into this small subset of $(\mathbb{S}^1)^{\mathcal{N}_n}$
- Or there is full correlation between $\mathcal{N}_n^A$ and $\exp(\mathcal{N}_n^{1-\varepsilon})$ but no replication.