# Part II

# Multiscale Analysis of Movies for Depth Recovery

# Chapter 9

# Introduction

## 9.1 The depth recovery problem

How can one establish a tridimensional map of a land area ? How can the tridimensional structure of a given object be measured ? How can one make a mobile robot perceive the geometry of an unknown environment ? All these problems are in fact the same : recovering the 3D-structure of a scene (land, object, environment) that can be observed. This problem of structure recovery has motivated many researches for the last twenty years, and multi-image analysis has been quickly identified as the most promising technique. Special devices like laser telemeters have sometimes been used, but for the time being their efficiency seems limited to very particular applications. As regards multi-image analysis, it is based on a simple geometric observation that everybody made once when looking through the side window of a car or a train : when one observes the landscape, the nearest objects "move" quicker than the farthest ones as the vehicle goes forward. Human stereo-vision is based on the same principle : between two observations from slightly different points of view (the two eyes), the relative positions of objects change according to their distance to the observer.

Inspired by human vision, researchers have studied in detail the technique of stereo vision analysis in the last two decades, in particular in association with edge-matching techniques. The principle is simple : the computer gets two pictures of the same scene from two cameras, then it detects on both images some features, for example, edges given by brisk contrast changes along straight lines. Last, it tries to match these edges (that is to say, it tries to associate each edge of the first image to its corresponding edge in the second image), and finally it recovers their depth by analyzing their relative position between the two images. This technique, after a certain success in the beginnings, finally appeared as insufficient for several reasons.

First, a simple analysis proves that the precision obtained in the determination of the depth is better when the cameras are far from each other, whereas the matching process is easier when they are close to each other. This incompatibility forced people to find a compromise between precision and robustness.

Another problem with edge-matching techniques is that they are more or less limited to artificial environments, because they require scenes with strongly-determined edges. In the case of natural textured scenes (e.g. a grass field), they are inefficient, and it can be a real problem to find alternative features to match.

Although edge-matching techniques were still receiving a lot of attention, some researchers tried to overcome the incompatibility between robustness and accuracy by considering whole sequences of images instead of only two images : the question of "depth from motion" was born. Even if the key to depth recovery is the same as to stereovision (analysis of the relative position of scene objects), using a large number of images appeared to bring great improvements. Of course, such a point of view was possible thanks to the increasing power of computers, both in storage capacity and in computation speed. Indeed, it is important to notice that a reasonable sequence of images (say 100 images of size 512x512 in 256 colors) represents 25 Mo of memory, which can be analyzed in a few minutes by a good workstation (for a simple algorithm). With 50 frames per second, this means that real-time movie analysis cannot be performed by now unless massive parallel machines are used.

The "depth from motion" problem (also called "structure from motion") was investigated mainly in two different ways. The first and probably most natural way is a generalization of stereovision techniques : the idea is to track robust features (edges, corners, ...) in the successive images and to deduce their depth from their velocity. This kind of method (see [35] for example) is only efficient for a certain kind of scene (typically, a high-contrasted artificial scene), due to the necessary use of edge-detection (or more generally, feature-detection) techniques.

The second approach for "structure from motion" was inspired by the classical Lagrangian formulation of the problem. It is based on the following Lambertian assumption : the color of a physical point does not depend on the point of view it is observed from. This assumption implies the famous "Motion Constraint Equation", which determines on the image sequence what is called the optical flow : this is simply the apparent velocity flow induced in the sequence of images by the apparent movement of the scene (induced itself by the camera movement). Numerous techniques have been developed in order to determine optical flow, but their efficiency is still debatable because of the stringent hypotheses they rely on (see [11] or [62] for detailed studies). In fact, the main difficulty of the general "structure from motion" problem in its Lambertian approach is that the system produced by the Motion Constraint Equation is under-determined : there are more unknowns than scalar equations. Even worse, the optical flow is not sufficient to recover the depth of objects for a general camera movement. Researchers tried to overcome the difficulty by writing regularity constraints, but this only brought partial solutions (or partial failures, depending on the point of view). In this context, the concept of active vision emerged (see [1]) : *"Most classical ill-posed problems of image sequence processing become well-posed and robust when the processing system controls the motion of the camera"*. Of course, such an assumption is not always relevant, for most image sequence analyzers are not real-time processes. However, the weakened and less restrictive assumption of a known camera movement (pre-determined or not) seems to be a good compromise : this will be our point of view.

During this study, we shall consider image sequences produced by a moving camera looking at a *fixed* scene (i.e. with no moving objects[1]). In addition, we shall make the assumption that the camera horizontal plane is fixed. This means that the optical axis of the camera and the horizontal axis of the image plane[2] remain in a fixed plane. In order to check that this condition is not too restrictive, we give some examples of camera movements which satisfy this assumption.

1. **Pure translation motion with transversal observation.**
   The camera path is a straight line parallel to the horizontal axis of the camera, and the optical axis remains orthogonal to this line (see Figure 9.1). This situation happens with a camera looking through the side window of a moving vehicle, to go back to our first example. This motion also occurs when an observation plane flies over a region at constant altitude with the camera optical axis pointing downwards[3]. Solving the depth recovery problem in this case enables to establish a 3D-map of the region which has been flown over. This camera movement will be our reference framework in the following.

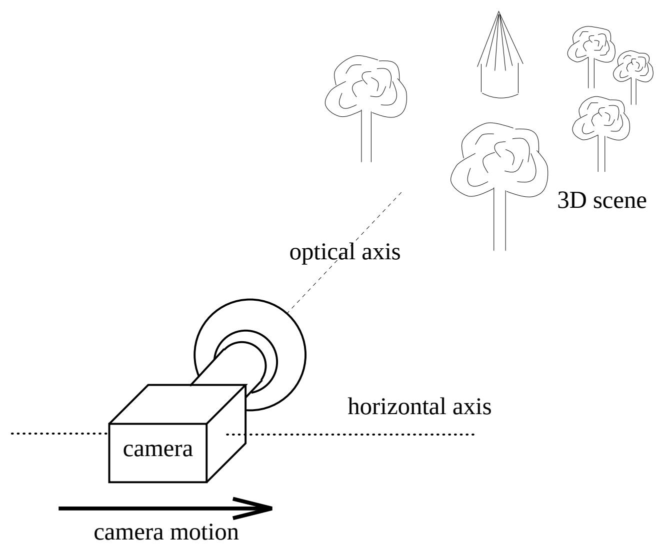


Figure 9.1: Pure translation motion.

2. **Circular motion.**

[1]Notice that it is equivalent to suppose that the camera is fixed and the whole scene has a rigid motion.
[2]The image plane, also called retinal plane or focal plane, is the plane where the physical image is produced by the optical lens system of the camera.
[3]However, we shall see later that our study can be adapted when the altitude of the plane varies with time or when the camera is not exactly pointing downwards.

This kind of motion is more adapted to the determination of the 3D-structure of a given object. The camera path is a circle, and the camera optical axis is constrained to point towards the center of this circle (see Figure 9.2). This motion also naturally occurs for non-geostationary satellites.

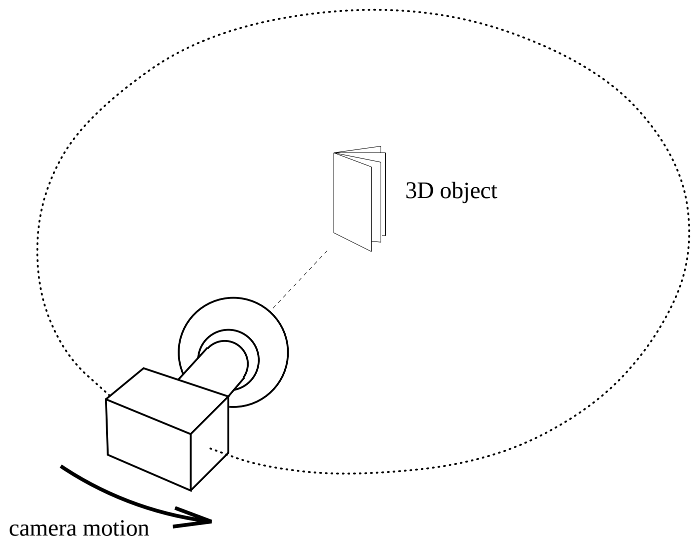


Figure 9.2: Circular motion.

3. **"Radar" motion**
   The camera has a pure rotational motion, and the optical axis remains orthogonal to the rotation axis (see Figure 9.3).

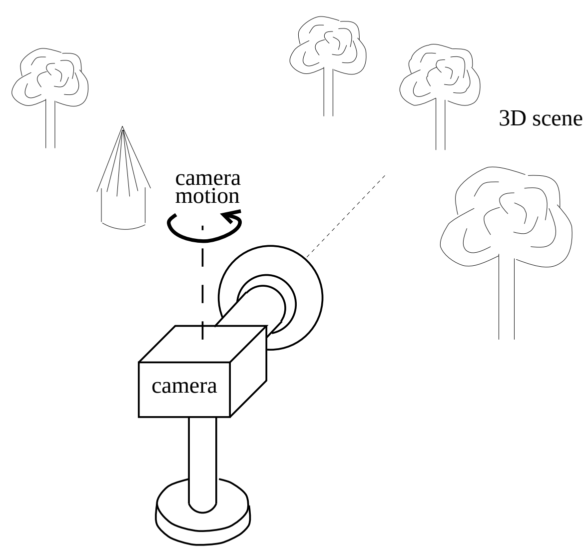


Figure 9.3: "Radar" motion.

The important aspect of the assumption we make on the camera movement is that it constrains the apparent movement of objects to be horizontal in the image plane. The three examples we gave prove that it is not too restrictive when the camera motion can be controlled. It has often been used in previous works (see [13] and [57] for example).

## 9.2 Geometric framework

We now come to more precise definitions and notations.

Consider a surface $\Sigma$ of $\mathbb{R}^3$ represented by the graph of the depth function $Z(X, Y)$. Suppose that $\Sigma$ is observed under a perspective projection[4] by a camera centered in $(C, 0, 0)$, with focal length $a$ and an optical axis directed by the $Z$ axis (see Figure 9.4). Each point $M = (X, Y, Z(X, Y))$ of $\Sigma$ is projected on the image plane $\Pi : Z = a$ into $P = (x, y) = \pi(M)$ defined by

$$\begin{cases} x = a\dfrac{X - C}{Z(X,Y)} \\ \\ y = a\dfrac{Y}{Z(X,Y)} \end{cases} \tag{9.1}$$

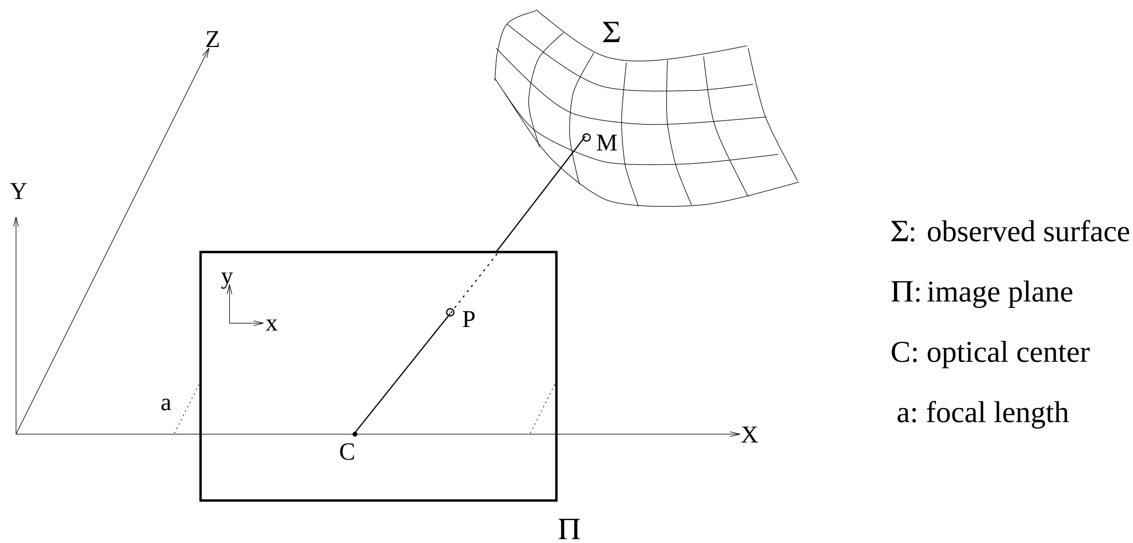


Figure 9.4: Scene geometry

Conversely, given a point $P$ of the image plane, we can define $\mu(P) \in \Sigma$ as the closest point to $P$ on the half line $[CP)$, when it exists. Thus, $\mu$ is a right inverse of $\pi$ since $\pi \circ \mu$ is the identity map of $\pi(\Sigma)$.

Now, if $\Sigma$ is a Lambertian surface characterized by its luminance $U(M)$, the camera produces the intensity image $u : P \mapsto U(\mu(P))$, up to an increasing rescaling depending on the intensity calibration of the camera. Notice that when the half line $[CP)$ intersects $\Sigma$ more than once, an **occlusion** arises, and only the nearest point (i.e. $\mu(P)$) is observed, the other ones being masked by it.

[4]This model of projection holds for classical "pinhole" cameras.

We extend this to the case when the camera is moving along the $X$ axis, the optical center following the path $(C(\theta), 0, 0)$, where $\theta$ is the time variable. This way, we define the maps $\pi_\theta : \Sigma \mapsto \Pi$ and $\mu_\theta : \Pi \mapsto \Sigma$, and the image $u : P \mapsto u(P)$ becomes a **movie** $u(P, \theta) = U(\mu_\theta(P))$, that is to say a continuous sequence of images regarded as a scalar map defined on a subset of $\mathbb{R}^3$.

The aim of our study is to compute the geometry of $\Sigma$ — its observed part actually — from the redundant information contained in the movie $(x, y, \theta) \mapsto u(x, y, \theta)$, knowing that it should satisfy the fundamental equation

$$u\left(a\frac{X - C(\theta)}{Z(X,Y)}, a\frac{Y}{Z(X,Y)}, \theta\right) = U(X, Y). \tag{9.2}$$

## 9.3 Velocity field

To simplify the problem, we shall now suppose that no occlusion appears (we shall discuss the general case later). Then, the relation $M \mapsto P$ is bijective, that is to say we have $\mu = \pi^{-1}$ on $\pi(\Sigma)$. This induces a bijective relation between the scalar **image maps** $f : \Pi \times \mathbb{R} \to \mathbb{R}$ and their corresponding **scene maps** $F : \Sigma \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(M, \theta) = f(\pi_\theta(M), \theta) = f(P(\theta), \theta).$$

Consider now a point $M$ of $\Sigma$. Projected on the movie, this point describes the movement $P(\theta) = \pi_\theta(M)$, whose velocity can be determined from Equation 9.1 :

$$\frac{dP}{d\theta} = (\frac{dx}{d\theta}, \frac{dy}{d\theta}) \quad \text{with} \quad \frac{dx}{d\theta} = -\frac{aC'(\theta)}{Z(X,Y)} \quad \text{and} \quad \frac{dy}{d\theta} = 0,$$

$C'$ meaning the derivative of $C$. Following this idea, we can define the derivative of an image map $f$ along the real movement by

$$\frac{\partial F}{\partial \theta} = \frac{d}{d\theta} f(P(\theta), \theta) = \prec Df, \frac{dP}{d\theta} \succ + \frac{\partial f}{\partial \theta} = \frac{dx}{d\theta} \cdot \frac{\partial f}{\partial x} + \frac{\partial f}{\partial \theta}.$$

In particular, if Equation 9.2 is satisfied, the derivative of $u$ along the movement must be zero, because the corresponding scene function $U(M)$ does not depend on $\theta$. This implies a specific formulation of the Motion Constraint Equation,

$$\frac{dx}{d\theta} \cdot \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \theta} = 0. \tag{9.3}$$

From this equation, it is natural to define the apparent velocity field of the movie by

$$v := -\frac{\frac{\partial u}{\partial \theta}}{\frac{\partial u}{\partial x}}, \tag{9.4}$$

when $\frac{\partial u}{\partial x} \neq 0$, remembering that if a scene interpretation exists (i.e. if Equation 9.2 is satisfied), we have

$$v(x, y, \theta) = -\frac{aC'(\theta)}{Z(X,Y)} \tag{9.5}$$

everywhere $v$ is defined (i.e everywhere $\frac{\partial u}{\partial x} \neq 0$).

Following this idea, we define the **total derivative** of a scalar image map $f : \Pi \times \mathbb{R} \to \mathbb{R}$ as

$$\frac{\mathrm{D}f}{\mathrm{D}\theta} = v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \theta}.$$

This is exactly the Lie derivative of $f$ along the apparent movement vector $\xi = (v, 0, 1)$. When a scene interpretation is known, it can be identified as the time derivative of the scene map associated to $f$. The importance of this total derivative operator will appear later.

## 9.4 Depth recovery

Theoretically, it is possible to estimate the apparent velocity field $v$ using Equation 9.4, and then to recover the depth $Z$ by identifying $v$ with the real velocity in Equation 9.5. This way, choosing a fixed value of $\theta$, we can hope to associate to any point $P = (x, y, \theta)$ of the image plane where the apparent velocity is defined and nonzero, the point $M = \mu_\theta(P)$ of $\Sigma$ defined by

$$M = \left( C(\theta) - \frac{C'(\theta)x}{va}, -\frac{C'(\theta)y}{va}, -\frac{C'(\theta)}{v} \right).$$

If $C(\theta)$, $C'(\theta)$ and $a$ are not known, the structure of $\Sigma$ is recovered up to a linear transformation of the kind

$$(X, Y, Z) \mapsto (\alpha X + \beta, \alpha Y, \gamma Z).$$

In practice, several difficulties appear when one tries to recover the geometry of $\Sigma$ directly. The first one occurs in the computation of $v$ from Equation 9.4. Indeed, it is impossible to obtain good estimations of the time derivative $\frac{\partial u}{\partial \theta}$ using finite difference methods. The reason is that most digital movies have a too large time sampling step, inasmuch as the number of images per second produced in the sampling process is too small compared to the quick change of scene details. In other words, the Nyquist limit is generally exceeded during the sampling process, simply because most acquisition systems (cameras, camescopes, ...) sample each image independently without first processing a time frequency cutoff[5]. Hence, Shannon's Theorem does not apply any more and common approximations cannot be used to estimate time derivatives. As concerns the spatial derivative $\frac{\partial u}{\partial x}$, its estimation hardly makes sense for textured areas, because of the quick changes in the intensity. For areas where the intensity takes a constant (or quasi-constant) value, the estimation of $v$ becomes very sensitive to noise and quantization, since the almost-zero quantity $\frac{\partial u}{\partial x}$ appears in the denominator of $v$.

The "classical" method to overcome this kinds of problem is to apply a linear spatio-temporal smoothing filter to the movie (see [13] for example), which can be seen as a (post-sampling) low-pass filter. Such a kind of isotropic diffusion has disastrous effects on non-smooth details like

[5] In fact, this is not really a bad thing since the non-continuous structure of images due to the presence of occlusions makes the classical sampling theory inadapted.

edges or textured areas. Like all linear filters, it is not adapted to the structure of images which result more from the superimposition of occluding objects than from the addition of weighted harmonics (see [21]).

Another problem appearing in the naive reconstruction process we just described is that two determinations of $\Sigma$ made from derivative estimations at different times $\theta_1$ and $\theta_2$ may produce slightly different results in practice, because real movies are not exactly time-coherent. This is a very important problem since, as we saw in the introduction, the large number of images is supposed to guarantee robustness and accuracy in the depth recovery.

All these remarks lead to think that the depth recovery must be achieved on a sort of ideal movie, for which the computation of $v$ can be made accurately and for which the depth interpretation of the scene remains the same at any time. One can reasonably hope to obtain such an ideal movie from a raw one thanks to the redundancy of the information spread among all images. In the following study, we shall see that such a transformation is possible, and that it can be obtained systematically by using an axiomatic formulation of the problem (Chapter 10). This transformation can be formalized by a non-linear diffusion equation along the movement field, which appears to have interesting properties (Chapter 11, 12). In Chapter 13, we provide a numerical algorithm, easily implementable —even on parallel machines—, as well as conclusive experiments on two classical natural movies. To conclude in Chapter 14, we generalize our study to more general camera motions and highlight further axes of extension.

# Chapter 10

# Axiomatic formulation

In this chapter, we devise a multiscale analysis of movies devoted to the depth recovery by using an axiomatic formulation. Such a methodology is not new : it has been successfully applied in [4] and in [26] to find the Affine Scale Space as the optimal way (in a certain sense) to simplify images and shapes. After making clear requirements, we establish a uniqueness result for our model : there is only one analysis of movies compatible with the depth recovery.

Let us first define some notations. Given an open or closed subset $\Omega$ of $\mathbb{R}^n$, $C^n(\Omega)$ means the space of continuous maps $u : \overline{\Omega} \to \mathbb{R}$ of class $C^n$ on $\Omega$. As usual, $\overline{\Omega}$ means the topological closure of $\Omega$ in $\mathbb{R}^n$. We shall also write $\mathcal{S}(\mathbb{R}^3)$ to denote the set of real symmetric 3x3 matrices.

As we saw previously, a movie is a real-valued map $u$ defined on a subset of $\mathbb{R}^3$, the value $u(x, y, \theta)$ representing the light intensity at a point $(x, y)$ of the plane at time $\theta$. The natural domain for a digital movie is $[x_1, x_2] \times [y_1, y_2] \times [\theta_1, \theta_2]$, but we shall see that it is simpler and more logical to suppose that a movie is defined on $\mathbb{R}^2 \times \overline{I}$, with either $I = ]\theta_1, \theta_2[$ or $I = S^1$ (case of a time-periodic movie).

We recall that a multiscale analysis is a family of operators $(T_t : \mathcal{M} \to \mathcal{M})_{t \geqslant 0}$, $t$ representing the scale of analysis. Here, $\mathcal{M}$ is a movie space, that is to say a space of continuous real-valued maps defined on $\mathbb{R}^2 \times \overline{I}$. The choice of $\mathcal{M}$ will become natural later, but is not necessary for the time being since we only want to find constraints on $(T_t)$. However, because of the singularity which appears in the computation of the velocity field when the partial derivative $u_x$ vanishes ($u_x$ is a short notation for $\frac{\partial u}{\partial x}$), we shall suppose in the following that for any $n \geqslant 1$, the space

$$\mathcal{M}^n = \{u \in \mathcal{M} \cap C^n(\mathbb{R}^2 \times \overline{I}, \mathbb{R});\ \forall \boldsymbol{z} \in \mathbb{R}^3, u_x(\boldsymbol{z}) \neq 0\}$$

is nonempty, and that given $(\lambda, \boldsymbol{p}, A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathcal{S}(\mathbb{R}^3)$, it is possible to find $u \in \mathcal{M}^2$ such that

$$u(\boldsymbol{0}) = \lambda, \quad Du(\boldsymbol{0}) = \boldsymbol{p} \quad \text{and} \quad D^2u(\boldsymbol{0}) = A.$$

## 10.1 Architectural axioms

In the spirit of [4], we first constrain our multiscale analysis to satisfy some architectural axioms :

- **[Recursivity]** : $T_0 = Id$ and $\forall t, t' > 0, \quad T_{t+t'} = T_{t'} \circ T_t$.

- **[Local Comparison Principle]** : if $u < \tilde{u}$ on $B(\boldsymbol{z}, r)$, then $T_t u(\boldsymbol{z}) \leqslant T_t \tilde{u}(\boldsymbol{z})$ for $t > 0$ small enough.

- **[Regularity]** : if $u$ is a quadratic form (that is, $u(\boldsymbol{z}) = [A](\boldsymbol{z}, \boldsymbol{z}) + \prec \boldsymbol{p}, \boldsymbol{z} \succ + \lambda$ where $A$ is a symmetric 3x3 matrix ($[A]$ being the associated bilinear map), $\boldsymbol{p}$ a 3-dimensional vector and $\lambda$ a given constant), then

$$\lim_{t \to 0} \frac{T_t u - u}{t}(\boldsymbol{z}) = F(A, \boldsymbol{p}, \lambda)$$

and $F$ depends continuously on $A$ when $p_1 \neq 0$ ($p_1$ being the component of $\boldsymbol{p}$ along the $x$ coordinate).

The **[Recursivity]** axiom constrains the multiscale analysis to have a semi-group structure. If the scale $t$ is discretized, this means that the analysis is obtained at scale $n$ by iterating $n$ times a fixed filter. This axiom can be weakened in

**[Pyramidal Architecture]** : $\forall t, h, \exists\, T_{t+h,t}, \quad T_{t+h} = T_{t+h,t} \circ T_t$.

However, we checked that under this hypothesis the final classification remains the same up to a rescaling (as it has been proved in [4] for the affine scale space). This is the reason why we directly assume that $(T_t)$ is a semi-group.

The **[Local Comparison Principle]** axiom is very important : it prevents the multiscale analysis from creating new details in the analyzed movie as the scale increases. It also guarantees the stability of associated numerical algorithms.

The **[Regularity]** axiom also contains the classical **[Translation Invariance]** axiom, which states that the multiscale analysis does not depend on the origin of space and time coordinates. When $I = ]\theta_1, \theta_2[$, the classical formulation of **[Translation Invariance]** is not possible any longer because the domain is not translation-invariant.

These axioms can be found in the axiomatic characterization of the affine morphological scale space for example ; only the **[Regularity]** axiom has been adapted to the depth recovery problem. Please refer to [40] for complete discussion.

The classification starts with the following theorem.

**Theorem 8** *A multiscale analysis $T_t : u_0(\cdot) \mapsto u(\cdot, t)$ satisfying **[Recursivity]**, **[Local Comparison Principle]** and **[Regularity]** can be described by a partial differential equation of the kind*

$$\frac{\partial u}{\partial t} = F(D^2 u, Du, u) \tag{10.1}$$

*submitted to initial condition* $u(\cdot, 0) = u_0$. *Moreover,* $F$ *is elliptic (that is to say nondecreasing with respect to its first argument for the usual order on 3x3 symmetric matrices), and continuous with respect with its first argument at any point where* $u_x \neq 0$.

The proof of an equivalent theorem can be found in [40] for example. The existence of $F$ is a direct consequence of the **[Regularity]** axiom. The fact that the evolution is given by a PDE of order two (and not more) results from the **[Local Comparison Principle]** axiom, as well as the ellipticity of $F$.

Notice that Equation 10.1 makes sense (in terms of existence and unicity of solutions) according to the theory of viscosity solutions (see [27]), provided that the singularity $u_x = 0$ is not involved. This point will become clearer in the next chapter. By now, the only important point is that Equation 10.1 is satisfied in the classical sense by $u$ at any point where $u$ is $C^2$ and $u_x \neq 0$.

## 10.2 Specific axioms

We now come to specific axioms with respect to the depth recovery problem. First, remember that when $u \in \mathcal{M}^n$ $(n \geqslant 1)$, the apparent velocity field operator is well defined by

$$v[u] = -\frac{u_\theta}{u_x}.$$

Since we are interested in the apparent velocity field, it seems natural that our analysis focuses mainly on this datum. In that sense, it is rather natural to constrain the analysis to commute with operators that preserve the apparent velocity field. This justifies the following axiom.

- **[v-Compatibility]**: For any $h : \mathbb{R}^4 \mapsto \mathbb{R}$, if

$$\forall u \in \mathcal{M}^1,\ R_h u \in \mathcal{M}^1 \quad \text{and} \quad v[R_h u] = v[u], \quad \text{with} \quad R_h u(x,y,\theta) = h\left(u(x,y,\theta), x, y, \theta\right),$$

then

$$\forall t,\ T_t \circ R_h = R_h \circ T_t.$$

This axiom implies two weaker axioms, obtained for specific choices of $h$.

◇ **[Strong Morphological Invariance]**: For any monotone scalar map $g$,

$$\forall u \in \mathcal{M}, \forall t,\ T_t g(u) = g(T_t u).$$

◇ **[Transversal Invariance]**: For any nonvanishing map $g$,

$$\forall u \in \mathcal{M}, \forall t,\ T_t(g(y) \cdot u) = g(y) \cdot (T_t u).$$

The first one is obtained by choosing $h(u,x,y,\theta) = g(u)$. It is a strong formulation of the morphological invariance, because $g$ can be decreasing as well as increasing. In fact, this axiom is equivalent to the classical **[Morphological Invariance]** axiom plus the **[Contrast reversal]** axiom. The second one, obtained with $h(u, x, y, \theta) = g(y) \cdot u$, is a kind of morphological invariance in the direction transversal to the movement. Notice that we supposed implicitely that $\mathcal{M}$ is stable under the operations $u \mapsto g \circ u$ and $u \mapsto g(y) \cdot u$. Following [40], we also constrain the analysis to commute with the superimposition of any uniform movement of the camera.

- **[Galilean Invariance]**:

$$\forall \alpha \in \mathbb{R},\ \forall u \in \mathcal{M},\ \forall t,\ T_t(u \circ B_\alpha) = (T_t u) \circ B_\alpha, \quad \text{with} \quad B_\alpha(x, y, \theta) = (x - \alpha\theta, y, \theta).$$

Last, we would like the analysis not to depend on the focal length of the camera (the $a$ variable in the previous chapter). This can be translated into a commutation with spatial homothetic transformations.

- **[Zoom Invariance]**:

$$\forall \lambda \neq 0,\ \forall u \in \mathcal{M}, \forall t,\ T_t(u \circ H_\lambda) = (T_t u) \circ H_\lambda, \quad \text{with} \quad H_\lambda(x, y, \theta) = (\lambda x, \lambda y, \theta).$$

## 10.3 Fundamental equation

We now prove that the set of axioms we constrained the multiscale analysis to satisfy is sufficient to restrain the possible analyses to one candidate only[1]. We shall prove later that this candidate is actually a solution.

**Theorem 9** *There exists at most one multiscale analysis of movies defined on $\mathcal{M}^2$ satisfying the architectural axioms plus **[v-Compatibility]**, **[Galilean Invariance]** and **[Zoom Invariance]**. It must be described by the partial differential equation*

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x} u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}. \tag{10.2}$$

**Remark 1 :** For the time being, Equation 10.2 is defined in the classical sense for $u(\cdot, t) \in \mathcal{M}^2$. In fact, we shall see in the next chapter how we can define weak solutions of Equation 10.2 that are not in $\mathcal{M}^2$ but only continuous.

**Remark 2 :** Equation 10.2 can be rewritten into

$$u_t = u_{\xi\xi} \quad \text{with} \quad \xi = (-\frac{u_\theta}{u_x}, 0, 1) \quad \text{and} \quad u_{\xi\xi} = [D^2 u](\xi, \xi),$$

[1] Of course, the identity operator is irrelevant here.

which means an anisotropic diffusion of $u$ along the movement direction. The apparent acceleration in the movie can be defined by

$$? = \frac{\mathrm{D}v}{\mathrm{D}\theta} = v_\theta + v v_x,$$

which can be expanded in

$$? = -\frac{1}{u_x}\left(u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\right) = -\frac{u_{\xi\xi}}{u_x}.$$

Hence, Equation 10.2 can also be rewritten into

$$u_t = -?\, u_x.$$

**Lemma 17** *For any multiscale analysis satisfying the architectural axioms and* ***[v-Compatibility]****, there exists a map* $F : \mathbb{R}^2 \mapsto \mathbb{R}$ *such that*

$$u_t = u_x F(?\,, v). \tag{10.3}$$

***Proof :***

Let us first make clear that the map $F$ we write here is not the map $F$ of Equation 10.1 : we simply use the same notation to avoid introducing too many symbols.

We are going to use the fact that the [**v-Compatibility**] axiom implies the simpler axioms [**Strong Morphological Invariance**] and [**Transversal Invariance**], as we noticed before.

Applying [**Strong Morphological Invariance**] for $g(u) = u+\lambda$ ($\lambda$ being an arbitrary constant) proves that $F$ cannot depend on $u$ in Equation 10.1, so that we have

$$\frac{\partial u}{\partial t} = G(D^2u, Du) \tag{10.4}$$

Now, the [**Transversal Invariance**] axiom states that for any nonvanishing function $g$ of class $C^2$,

$$\forall u \in \mathcal{M}^2, \forall y, \quad G(D^2(g(y)\cdot u), D(g(y)\cdot u)) = G(D^2u, Du). \tag{10.5}$$

Let $A = [a_{ij}] \in \mathcal{S}(\mathbb{R}^3)$, $\lambda \in \mathbb{R}$ and $\boldsymbol{p} = (p_i) \in \mathbb{R}^3$ such that $p_1 \neq 0$ (the coordinates $x, y, \theta$ will be indexed by $1, 2, 3$ in the following). By hypothesis on $\mathcal{M}^2$, we can build a movie $u \in \mathcal{M}^2$ such that

$$u(0,0,0) = \lambda, \qquad Du(0,0,0) = \boldsymbol{p}, \qquad \text{and} \qquad D^2u(0,0,0) = A.$$

Now, consider the vector $\boldsymbol{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, the projection matrix on the $(x,\theta)$ plane

$$Q_{y^\perp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the projection matrix on the line $\mathbb{R}\boldsymbol{y}$

$$Q_y = \boldsymbol{y} \otimes \boldsymbol{y} = I - Q_{y^\perp} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$I$ being the identity matrix of $\mathcal{S}(\mathbb{R}^3)$. Applying Equation 10.5 to $u$ in $(0,0,0)$, we obtain

$$G(g(0)A + g'(0)\boldsymbol{y} \otimes \boldsymbol{p} + g''(0)\lambda Q_y, g(0)\boldsymbol{p} + g'(0)\boldsymbol{y}) = G(A, \boldsymbol{p}).$$

If we choose $g(y) = 1 + y^2/2$, we get

$$\forall A, p, \lambda, \quad G(A + \lambda Q_y, \boldsymbol{p}) = G(A, \boldsymbol{p}), \tag{10.6}$$

and taking $\lambda = -a_{22}$ yields

$$\forall A, \boldsymbol{p}, \quad G(..., a_{22}, ...) = G(..., 0, ...),$$

where the two terms only differ in the $a_{22}$ variable. Hence, $G$ does no depend on $a_{22}$.

Now we are going to show that $G$ does not depend on $a_{12}$ and $a_{23}$ either, by using the **[Causality]** axiom, using a technique from Giga et Goto [37][2]. Let us define $A' = A - a_{22}Qy$ and for $\varepsilon > 0$,

$$I_\varepsilon = \varepsilon Q_{y^\perp} + \frac{a_{21}^2 + a_{23}^2}{\varepsilon} Q_y = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \frac{a_{21}^2 + a_{23}^2}{\varepsilon} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

The characteristic polynom of the matrix

$$A_\varepsilon = Q_{y^\perp} A' Q_{y^\perp} - A' + I_\varepsilon = \begin{pmatrix} \varepsilon & -a_{21} & 0 \\ -a_{21} & \frac{a_{21}^2 + a_{23}^2}{\varepsilon} & -a_{23} \\ 0 & -a_{23} & \varepsilon \end{pmatrix}$$

is

$$\det(xI - A_\varepsilon) = x(x - \varepsilon)\left(x - (\varepsilon + \frac{a_{21}^2 + a_{23}^2}{\varepsilon})\right).$$

As the eigenvalues of $A_\varepsilon$ are nonnegative, $A_\varepsilon$ is positive (for the usual order in $\mathcal{S}(\mathbb{R}^3)$), and symmetrically $A_{-\varepsilon}$ is negative, which yields

$$A' - I_\varepsilon \leqslant Q_{y^\perp} A' Q_{y^\perp} \leqslant A' + I_\varepsilon$$

But the **[Causality]** axiom implies (see [37])

$$\forall A, B, \boldsymbol{p}, \quad A \geqslant B \;\Rightarrow\; G(A, p) \geqslant G(B, \boldsymbol{p}),$$

[2] If we suppose that $G$ is differentiable, then this property follows immediatly. Indeed, the **[Causality]** axiom implies

$$\forall i, j, \;\; \det_{i,j}[D^2 G] = \frac{\partial G}{\partial a_{ii}} \frac{\partial G}{\partial a_{jj}} - \left(\frac{\partial G}{\partial a_{ij}}\right)^2 \geqslant 0$$

and since $\frac{\partial G}{\partial a_{22}} = 0$, we get $\frac{\partial G}{\partial a_{21}} = \frac{\partial G}{\partial a_{23}} = 0$

so that

$$\forall A, \boldsymbol{p}, \quad G(A' - I_\varepsilon, \boldsymbol{p}) \leqslant G(Q_{y^\perp} A' Q_{y^\perp}, \boldsymbol{p}) \leqslant G(A' + I_\varepsilon, \boldsymbol{p}).$$

Then, using Equation 10.6, we get

$$\forall A, \boldsymbol{p}, \quad G(A + \varepsilon I, \boldsymbol{p}) \leqslant G(Q_{y^\perp} A Q_{y^\perp}, \boldsymbol{p}) \leqslant G(A + \varepsilon I, \boldsymbol{p})$$

and taking the limit when $\varepsilon \to 0$, the continuity of $G$ implies

$$\forall A, \boldsymbol{p}, \quad G(A, \boldsymbol{p}) = G(Q_{y^\perp} A Q_{y^\perp}, \boldsymbol{p}),$$

which means that we can write

$$\forall A, \boldsymbol{p}, \quad G(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, p_1, p_2, p_3) = H(a_{11}, a_{13}, a_{33}, p_1, p_2, p_3).$$

Now, applying again the [**Transversal Invariance**] axiom to $H$, we obtain

$$\forall A, \boldsymbol{p}, g, y \quad H(a_{11}, a_{13}, a_{33}, p_1, p_2, p_3) = H(a_{11}, a_{13}, a_{33}, p_1, g'(y) p_2, p_3) \tag{10.7}$$

Choosing $p_2 = 1$ and $g(y) = 1 + y^2/2$ as before, Equation 10.7 yields

$$\forall A, p_1, p_3, y \quad H(a_{11}, a_{13}, a_{33}, p_1, 1, p_3) = H(a_{11}, a_{13}, a_{33}, p_1, y, p_3),$$

which proves that $H$ does not depend on its fifth argument $p_2$.

Now we use the [**Strong Morphological Invariance**] axiom. It has been proven (see [4] for example) that this axiom, in combination with the [**Causality**] axiom, forces the second order terms of the evolution to be of the kind $[D^2 u](\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the plane orthogonal to $Du$, written $(Du)^-$. Now, as we just saw, the [**Transversal Invariance**] axiom forbids any dependency on $y$, so that $\boldsymbol{a}$ and $\boldsymbol{b}$ must also belong to the $(\boldsymbol{y})^- = (x, \theta)$ plane. Finally, $\boldsymbol{a}$ and $\boldsymbol{b}$ must belong to the line $(Du)^- \cap (\boldsymbol{y})^- = (\xi)^-$, so that the only admissible second order term is $? = -\frac{1}{u_x}[D^2 u](\xi, \xi)$, up to a multiplicative first order term. Notice that ? is a morphological operator.

As regards the first order terms, the [**Transversal Invariance**] axiom forbids any dependency on $u_y$. Hence, as ? does not contain the $u_y$ term, $u$ must satisfy an evolution equation of the kind

$$u_t = F(?, u_x, u_\theta).$$

We rewrite this equation into

$$u_t = u_x G(?, v, u_x)$$

and apply the [**Strong Morphological Invariance**] axiom. Since ? and $v$ are morphological operators, it yields

$$\forall u, \ \forall \lambda \neq 0, \quad G(?, v, u_x) = G(?, v, \lambda u_x). \tag{10.8}$$

For any $(\alpha, \beta) \in \mathbb{R}^2$, we consider a movie $u \in \mathcal{M}^2$ such that

$$u(x, y, \theta) = \frac{\alpha}{2}\theta^2 + x - \beta\theta$$

in a vicinity of $(x, y, \theta) = \mathit{0}$. We have $u_x(\mathit{0}) = 1$, $v(\mathit{0}) = \beta$ and $?(\mathit{0}) = \alpha$ so that Equation 10.8 can be rewritten into

$$\forall \alpha, \beta,\ \forall \lambda \neq 0,\ \ G(\alpha, \beta, 1) = G(\alpha, \beta, \lambda),$$

which means that $G$ does not depend on its third argument (notice that $G$ does not need to be defined when $u_x = 0$). As a consequence, we can write

$$u_x = u_x\, F(?\,, v)$$

as announced. □

**Remark :** We proved that the **[v-Compatibility]** axiom, in association with the architectural axioms, forbids any dependency of the evolution on $y$. In other words, the sliced images $(x, \theta) \mapsto u(x, y, \theta)$ (with $y$ fixed) are processed independently. In the following, we shall often ignore the $y$ coordinate and we shall write $u(x, \theta)$ instead of $u(x, y, \theta)$, the $y$ variable being supposed fixed.

**Lemma 18** *A multiscale analysis satisfying the architectural axioms plus* ***[v-Compatibility]*** *and* ***[Galilean Invariance]*** *can be written*

$$u_t = u_x F(?) \tag{10.9}$$

***Proof :***

Since the multiscale analysis commutes with the operator

$$B_\alpha : (x, y, \theta) \mapsto (x - \alpha\theta, y, \theta),$$

we have

$$\frac{\partial}{\partial t}(u \circ B_\alpha) = \frac{\partial u}{\partial t} \circ B_\alpha.$$

Writing $\tilde{u} = u \circ B_\alpha$ yields

$$\begin{aligned}
\tilde{u}_x &= \frac{\partial}{\partial x} u(x - \alpha\theta, \theta) = u_x \circ B_\alpha \\
\tilde{u}_\theta &= \frac{\partial}{\partial \theta} u(x - \alpha\theta, \theta) = (u_\theta - \alpha u_x) \circ B_\alpha \\
\tilde{v} &= -\frac{\tilde{u}_\theta}{\tilde{u}_x} = v \circ B_\alpha + \alpha \\
\tilde{?} &= \frac{\mathrm{D}\tilde{v}}{\mathrm{D}\theta} = \tilde{v}_\theta + \tilde{v}\tilde{v}_x = (v_\theta - \alpha v_x + (v + \alpha) v_x) \circ B_\alpha = ? \circ B_\alpha.
\end{aligned}$$

From Lemma 17 we know that $u_t = u_x F(?\,, v)$. Hence,

$$\forall u, \alpha, \quad u_x F(?\,, v+\alpha) = u_x F(?\,, v),$$

so that $F$ does not depend on its second argument. □

**Lemma 19** *A multiscale analysis satisfying the architectural axioms plus **[v-Compatibility]** and **[Zoom Invariance]** can be written*

$$u_t = \begin{cases} u_\theta F(\frac{?}{v}) \text{ if } u_\theta \neq 0, \\ \\ a u_x \text{ if } u_\theta = 0. \end{cases} \tag{10.10}$$

***Proof :***

We proceed as for Lemma 18 : writing $\tilde{u} = u \circ H_\lambda$ with $H_\lambda : (x, y, \theta) \mapsto (\lambda x, \lambda y, \theta)$, we get

$$\begin{aligned} \tilde{v} &= -\frac{\tilde{u}_\theta}{\tilde{u}_x} = -\frac{u_\theta}{\lambda u_x} \circ H_\lambda = \frac{v}{\lambda} \circ H_\lambda \\ \tilde{?} &= \tilde{v}_\theta + \tilde{v}\tilde{v}_x = (\frac{v_\theta}{\lambda} + \frac{v}{\lambda}\frac{\lambda v_x}{\lambda}) \circ H_\lambda = \frac{?}{\lambda} \circ H_\lambda \end{aligned}$$

We can write Equation 10.3 as

$$u_t = u_x F(?\,, v) = u_\theta G(\frac{?}{v}, v)$$

everywhere $u_\theta \neq 0$, and since the evolution commutes with $H_\lambda$, we have

$$\forall u, \lambda, \quad u_\theta G(\frac{?}{v}, v) = u_\theta G(\frac{?}{v}, \frac{v}{\lambda}).$$

Taking the limit $\lambda \to \infty$ proves that $G$ cannot depend on its second argument. Besides, everywhere $u_\theta = 0$ we have

$$\forall u, \lambda, \quad u_x F(?\,, 0) = u_x F(\frac{?}{\lambda}, 0),$$

so that $F(?\,, 0) = F(0, 0)$. □

***Proof of Theorem 9 :***

If a multiscale analysis satisfies the axioms of Theorem 9, the corresponding evolution equation can be written in both forms given in Equation 10.9 and Equation 10.10. But the only common case is

$$u_t = -u_x ? = u_\theta \frac{?}{v} = u_{\xi\xi},$$

which is the announced equation. □

Conversely, we have to check that it is possible to define from Equation 10.2 a multiscale analysis of movies satisfying the previous axioms. This is the aim of the next chapter.

# Chapter 11

# The Depth-Compatible Multiscale Analysis

In this chapter, we give a rigorous definition for the DCMA Equation[1]

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}. \qquad (DCMA)$$

We define classical and weak solutions, and we establish uniqueness and existence theorems in both cases. We also establish the link with the theory of viscosity solutions of second order partial differential equations.

## 11.1 Classical solutions of the DCMA

For the reason we explained before, we forget the $y$ variable in the following, and a movie is defined on $\mathbb{R} \times \overline{I}$, with either $I = ]\theta_1, \theta_2[$ or $I = S^1$. In the space variable, a periodization has no meaning in terms of scene interpretation, so that we shall rather suppose that $u$ tends towards some constant when $x$ grows to infinity. Notice that such a condition is classical, even in a more restrictive formulation (e.g. $u$ *equals* a constant outside a compact set, see [31] for example).

**Definition 20** *For $c = (c^-, c^+) \in \mathbb{R}^2$ and $n \geqslant 0$, $\mathcal{C}_c^n$ is the space of movies $u \in C^n(\mathbb{R} \times I)$ such that*

$$\sup_{\theta \in \overline{I}} |u(-x,\theta) - c^-| + |u(x,\theta) - c^+| \to 0 \quad \text{as} \quad x \to +\infty. \qquad (11.1)$$

In all the following, we write $\Omega = \mathbb{R} \times I \times ]0, +\infty[$ ($\overline{\Omega}$ is the domain of movie analyses).

**Definition 21** *For $c \in \mathbb{R}^2$ and $n, p \geqslant 0$, $\mathcal{C}_c^{n,p}$ is the space of movie analyses $u \in C^0(\overline{\Omega})$ such that*

[1] The reason why we call this evolution equation DCMA (for Depth-Compatible Multiscale Analysis) will become clear in the next chapter.

1. $\sup_{\theta\in\overline{I}, t\leqslant R} |u(-x,\theta,t)-c^-|+|u(x,\theta,t)-c^+| \to 0$ as $x\to+\infty$,

2. *on* $\Omega$, $(x,\theta,t)\mapsto u(x,\theta,t)$ *is of class* $C^n$ *with respect to* $(x,\theta)$ *and* $C^p$ *with respect to* $t$.

When $c^-=c^+=0$, we shall say that $u$ is “null at infinity”.

Let us come back to our problem. We want to define classical solutions of Equation (DCMA). However, the space $\mathcal{M}^2$ we introduced in the axiomatic formulation is too restrictive, because of the condition $u_x\neq 0$. Indeed, this condition forces the partial maps $x\mapsto u(x,y,\theta)$ to be increasing or decreasing, which is not satisfactory, and prevents $u$ from satisfying Equation 11.1 with $c^+=c^-$ (this is the reason why we did not constrain $c^+=c^-$ in the previous definitions : since we want the axiomatic formulation to be relevant, the space $\mathcal{M}^2$ must be nonempty). For those reasons, we forget the condition $u_x\neq 0$ and write a degenerate formulation of Equation (DCMA) when $u_x$ vanishes.

**Example :** Consider $g\in C^2(\mathbb{R})$ such that $g(x)\to 0$ as $|x|\to+\infty$. We define the movie analysis $u:\mathbb{R}\times S^1\times[0,+\infty[\to\mathbb{R}$ by

$$u(x,\theta,t)=g(x-\theta^2-2t),$$

the representant of $\theta$ being taken in $[-\pi,\pi[$. Then, Equation (DCMA) is satisfied by $u$ at any point where $u_x\neq 0$, and when $u_x=0$ we have also $u_t=0$. This suggests a simple degenerate formulation of Equation (DCMA) when $u_x$ vanishes.

Incidentally, notice that $u\in\mathcal{C}_0^{2,2}$, but the condition

$$\sup_{\theta\in\overline{I}, t\geqslant 0} |u(x,\theta,t)|\to 0 \quad \text{as} \quad |x|\to+\infty$$

is not satisfied unless $g\equiv 0$. This is the reason why it is logical to consider the sup on $\{\theta\in\overline{I}, t\leqslant R\}$ in Condition 1 of Definition 21.

**Definition 22** *Given* $u_0\in\mathcal{C}_c^0$, *we say that* $u$ *is a classical solution of the DCMA associated to the initial datum* $u_0$ *if*

*(i)* $u\in\mathcal{C}_c^{2,1}$,

*(ii)* *on* $\Omega=\mathbb{R}\times I\times]0,+\infty[$,

$$\begin{cases} u_t=u_{\theta\theta}-2\dfrac{u_\theta}{u_x}u_{\theta x}+(\dfrac{u_\theta}{u_x})^2u_{xx} & \text{when} \quad u_x\neq 0,\\[2ex] u_t=0 \quad \text{when} \quad u_x=0. \end{cases}$$

*(iii)* $\forall(x,\theta,t)\in\partial\Omega,\ \ u(x,\theta,t)=u_0(x,\theta)$.

**Remark :** If $I = S^1$, $\partial I = \emptyset$ and the boundary condition (iii) means

$$\forall (x,\theta) \in \mathbb{R} \times \overline{I}, \quad u(x,\theta,0) = u_0(x,\theta).$$

If we choose to fix a time-boundary condition (i.e. $I = ]\theta_1, \theta_2[$) instead of a time-periodicity condition, (iii) also constrains

$$\forall (x,t) \in \mathbb{R} \times [0,+\infty[, \quad u(x,\theta_i,t) = u_0(x,\theta_i) \ \text{ for } \ i = 1,2$$

In order to state the uniqueness of solutions, we first establish a comparison principle.

**Lemma 20 (comparison principle)** *Suppose that $u$ and $\tilde{u}$ are two classical solutions of the DCMA associated to initial data $u_0$ and $\tilde{u}_0$ respectively. If $u_0 \leqslant \tilde{u}_0$, then $u \leqslant \tilde{u}$ on $\Omega$.*

***Proof :***

For $R > 0$, let us write

$$\varepsilon(R) = \sup_{|x| \geqslant R, \theta \in \overline{I}, t \leqslant R} u(x,\theta,t) - \tilde{u}(x,\theta,t).$$

Since $u$ and $\tilde{u}$ belong to $\mathcal{C}^{2,1}_c$ and $\mathcal{C}^{2,1}_{\tilde{c}}$, we have

$$\varepsilon(R) \to \max(c^- - \tilde{c}^-, c^+ - \tilde{c}^+) \quad \text{as} \quad R \to +\infty,$$

with $c - \tilde{c} \leqslant 0$ because $u_0 \leqslant \tilde{u}_0$. For $\alpha > 0$, we consider

$$\Lambda(x,\theta,t) = u(x,\theta,t) - \tilde{u}(x,\theta,t) - \alpha t.$$

On the compact set $K_R = [-R,R] \times \overline{I} \times [0,R]$, the continuous map $\Lambda$ attains its maximum value at a point $\boldsymbol{z}_0 = (x_0, \theta_0, t_0)$.

1. Suppose that

$$|x_0| < R, \quad \theta_0 \in I \quad \text{and} \quad t_0 \in ]0,R]. \tag{11.2}$$

In $\boldsymbol{z}_0$ we have

$$\Lambda_x = \Lambda_\theta = 0, \quad \Lambda_t \geqslant 0 \quad \text{and} \quad D^2\Lambda \leqslant 0.$$

This yields

$$Du(\boldsymbol{z}_0) = D\tilde{u}(\boldsymbol{z}_0), \tag{11.3}$$

$$u_t(\boldsymbol{z}_0) - \tilde{u}_t(\boldsymbol{z}_0) \geqslant \alpha, \tag{11.4}$$

$$\text{and} \quad D^2u(\boldsymbol{z}_0) \leqslant D^2\tilde{u}(\boldsymbol{z}_0), \tag{11.5}$$

the last inequality being meant for the usual order on symmetric 2x2 matrices.

1.a. If $u_x(\boldsymbol{z}_0) \neq 0$, then $\tilde{u}_x(\boldsymbol{z}_0) = u_x(\boldsymbol{z}_0) \neq 0$. Now recall that

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx} = F(D^2u, Du),$$

where $F$ is an elliptic operator, that is to say nondecreasing with respect to its first argument. Hence, Equations 11.3 and 11.5 imply $u_t(\boldsymbol{z}_0) \leqslant \tilde{u}_t(\boldsymbol{z}_0)$, which is in contradiction with Equation 11.4.

1.b. If $u_x(\boldsymbol{z}_0) = 0$, then $\tilde{u}_x(\boldsymbol{z}_0) = 0$, and since $u$ and $\tilde{u}$ are solutions of the DCMA, we have $u_t(z_0) = \tilde{u}_t(z_0) = 0$, which is a contradiction with Equation 11.4.

2. As a consequence of 1.a and 1.b, Assumption 11.2 is false and necessarily we have either $|x_0| = R$ or $\theta_0 \in \partial I$ or $t_0 = 0$. If $|x_0| = R$, then $\Lambda(x_0, \theta_0, t_0) \leqslant \varepsilon(R) + \alpha R$, while $\Lambda(x_0, \theta_0, t_0) \leqslant \alpha R$ when $\theta_0 \in \partial I$ or $t_0 = 0$. Consequently, we have

$$\max_{K_R} \Lambda \leqslant \max(0, \varepsilon(R)) + \alpha R,$$

and making $\alpha \to 0$ proves that

$$u \leqslant \tilde{u} + \max(0, \varepsilon(R)) \qquad \text{on} \quad \mathbb{R} \times \overline{I} \times [0, R].$$

Last, sending $R$ to infinity forces $\max(0, \varepsilon(R))$ to vanish and the proof is complete. □

**Corollary 9 (contraction property)** *If $u$ and $\tilde{u}$ are two classical solutions of the DCMA associated to the initial data $u_0$ and $\tilde{u}_0$, then*

$$\|u - \tilde{u}\|_\infty \leqslant \|u_0 - \tilde{u}_0\|_\infty.$$

***Proof :***

We simply need to notice that

$$u_0 - \|u_0 - \tilde{u}_0\|_\infty \leqslant \tilde{u}_0 \leqslant u_0 + \|u_0 - \tilde{u}_0\|_\infty,$$

and apply the comparison principle, remarking that if $u$ is a classical solution of the DCMA, so is $u + \lambda$ for any $\lambda \in \mathbb{R}$. □

**Corollary 10 (uniqueness)** *A classical solution of the DCMA associated to a given initial datum $u_0 \in \mathcal{C}_c^2$ is unique.*

The proof follows immediatly from Corollary 9.

In order to ensure the existence of classical solutions of the DCMA, we now restrain the space of initial data.

**Definition 23** *For $n \geqslant 1$, we write $\mathcal{V}_c^n$ the space of movies $u \in \mathcal{C}_c^n$ for which there exists a movie $v \in \mathcal{C}_0^{n-1}$ such that*

$$u_\theta + v u_x = 0 \quad \text{on} \quad \mathbb{R} \times \overline{I}. \tag{11.6}$$

*$v$ is called a velocity map of $u$.*

The space $\mathcal{V}_c^{n,p}$ is defined as elements of $\mathcal{C}_c^{n,p}$ admitting a velocity map $v \in \mathcal{C}_0^{n-1,p}$.

**Remark :** Consider a movie $u \in \mathcal{V}_c^n$. If $u_x(x,\theta) \neq 0$, $v(x,\theta)$ is uniquely determined because Equation 11.6 forces

$$v(x,\theta) = -\frac{u_\theta}{u_x}(x,\theta).$$

But as we noticed previously, $u_x(x,\theta)$ is forced to vanish at least once for any value of $\theta$, because

$$\lim_{|x|\to+\infty} u(x,\theta) = c.$$

When $u_x(x,\theta) = 0$, Equation 11.6 implies $u_\theta(x,\theta) = 0$, and if $n \geqslant 2$, differentiating Equation 11.6 with respect to $\theta$ and $x$ yields

$$u_{\theta\theta}(x,\theta) + v(x,\theta) u_{x\theta}(x,\theta) = 0 \tag{11.7}$$

and

$$u_{\theta x}(x,\theta) + v(x,\theta) u_{xx}(x,\theta) = 0. \tag{11.8}$$

We deduce from Equation 11.7 and 11.8 that $u_{\theta\theta} + 2v u_{\theta x} + v^2 u_{xx} = 0$ as soon as $u_x = 0$.

A consequence is that if $u \in \mathcal{V}_c^{2,1}$ is a classical solution of the DCMA, then any velocity map $v$ of $u$ satisfies on $\Omega$

$$\begin{cases} u_\theta + v u_x = 0 \\ u_t = u_{\theta\theta} + v u_{\theta x} + v^2 u_{xx}. \end{cases} \tag{11.9}$$

□

**Proposition 27 (existence)** *Given an initial datum $u_0 \in \mathcal{V}_c^n$ $(n \geqslant 2)$, there exists a unique classical solution of the DCMA, and it belongs to $\mathcal{V}_c^{n,n}$.*

***Proof :***

The existence will be a consequence of Lemma 22 (which follows), and the uniqueness follows from Corollary 10. □

We are going to build explicit solutions of the DCMA. The idea is to notice that the trajectories (i.e. the curves $x(\theta)$ along which $u$ is constant) are smoothed by the linear heat equation. For that purpose, we need to introduce the natural domain $I^\star$ for such trajectories. If $I = ]\theta_1, \theta_2[$ then $I^\star = I$, and if $I = S^1$, then $I^\star = \mathbb{R}$ (the natural injection $S^1 \hookrightarrow [0, 2\pi[ \subset \mathbb{R}$ being implicit). To simplify the notations, we suppose in the following that $0 \in \overline{I}$.

**Definition 24** *A map $\varphi \in C^n(\mathbb{R} \times I^\star)$ $(n \geqslant 0)$ is a **$\theta$-graph** of $u \in \mathcal{C}_c^n$ if*

1. *for any $\theta \in \overline{I^\star}$, the map $x \mapsto \varphi(x,\theta)$ is increasing and bijective*
2. *for any $(x,\theta) \in \mathbb{R} \times \overline{I^\star}$,*
$$u(\varphi(x,\theta),\theta) = u(x,0), \tag{11.10}$$
3. *for any $x \in \mathbb{R}$, $\varphi(x,0) = x$, and if $I = S^1$, then for any $(x,\theta) \in \mathbb{R} \times \overline{I^\star}$,*
$$\varphi(x,\theta+2\pi) = \varphi(\varphi(x,2\pi),\theta), \tag{11.11}$$
4. $\sup_{|x|\geqslant R, \theta \in \overline{I}} |\varphi_\theta(x,\theta)| \to 0 \quad \text{as} \quad R \to +\infty$ *(in a generalized sense if $n = 0$).*

**Remark :** Notice that in Condition 4, the sup is taken for $\theta \in \overline{I}$ and not for $\theta \in \overline{I^\star}$. If $n = 0$, the term $|\varphi_\theta(x,\theta)|$ must be replaced by

$$\limsup_{h\to 0} |\frac{\varphi(x,\theta+h) - \varphi(x,\theta)}{h}|.$$

A simple proof by induction establishes that when $I = S^1$, Equation 11.11 implies

$$\varphi(x,\theta+2\pi n) = \varphi(\varphi(x,2\pi n),\theta)$$

for any $(x,\theta,n) \in \mathbb{R} \times \overline{I^\star} \times \mathbb{N}$.

**Lemma 21** *A movie $u \in \mathcal{C}_c^n$ $(n \geqslant 2)$ belongs to $\mathcal{V}_c^n$ if and only if it admits a $\theta$-graph of class $C^n$.*

***Proof :***

1. Suppose that $u$ admits a $\theta$-graph of class $C^n$. Then, Condition 1 implies that the relation

$$v(\varphi(x,\theta),\theta) = \varphi_\theta(x,\theta) \tag{11.12}$$

defines a unique continuous map $v$ on $\mathbb{R} \times \overline{I}$ (if $I = S^1$, Equation 11.11 ensures the periodicity of $v$ in the $\theta$ variable). We can write

$$\forall (x,\theta) \in \mathbb{R} \times I,\ \forall h \in \mathbb{R},\ \ v(\varphi(x,\theta) + h\varphi_x(x,\theta) + o(h),\theta) = \varphi_\theta(x,\theta) + h\varphi_{\theta x}(x,\theta) + o(h).$$

Since $\varphi_x(x,\theta) > 0$ a.e. due to Condition 1, we deduce that $v$ is derivable with respect to $x$ and

$$\varphi_x(x,\theta)v_x(\varphi(x,\theta),\theta) = \varphi_{\theta x}(x,\theta).$$

A similar reasoning proves that $v$ is of class $C^{n-1}$. Differentiating Equation 11.10 with respect to $\theta$ yields

$$\forall (x,\theta) \in \mathbb{R} \times \overline{I}, \quad \varphi_\theta(x,\theta)u_x(\varphi(x,\theta),\theta) + u_\theta(\varphi(x,\theta),\theta) = 0,$$

so that $v$ is a velocity map of $u$ thanks to Equation 11.12.

Now let us write diam $(I)$ the diameter of $I$. Given $\varepsilon > 0$, Condition 4 ensures the existence of a $R > 0$ such that

$$\forall (x,\theta) \in \mathbb{R} \times \overline{I}, \qquad |x| \geqslant R \;\Rightarrow\; |\varphi_\theta(x,\theta)| \leqslant \varepsilon.$$

Hence, if $|x| \geqslant R' = R + \varepsilon \cdot \operatorname{diam}(I)$ we have

$$\varphi(x,\theta) = \varphi(x,0) + \int_0^\theta \varphi_\theta(x,\tau)d\tau \geqslant x - \varepsilon|\theta| \geqslant R$$

and consequently

$$\sup_{|x|\geqslant R', \theta \in \overline{I}} |v(x,\theta)| \leqslant \varepsilon.$$

It follows that $v \in \mathcal{C}_0^{n-1}$ and the same reasoning proves that $u$ is constant at infinity, so that $u \in \mathcal{V}_c^n$.

2. Conversely, if $u \in \mathcal{V}_c^n$, consider a velocity movie $v$ of $u$. Given $(x_0, \theta_0) \in \mathbb{R} \times \overline{I}$, there exists a unique solution $X \in C^n(\overline{I^\star})$ of the ordinary differential equation

$$\frac{dX}{d\theta}(\theta) = v(X(\theta), \theta) \tag{11.13}$$

submitted to the condition $X(\theta_0) = x_0$. Since $v \in \mathcal{C}_0^{n-1}$, $v$ is bounded, so that $X$ is defined on the whole interval $\overline{I^\star}$. Call $\varphi(x_0, \theta)$ the solution $X$ associated to $\theta_0 = 0$, and let $k = \operatorname{diam}(I) \cdot \|v\|_\infty$. Then

$$\sup_{|x|\geqslant R, \theta\in\overline{I}} |\varphi_\theta(x,\theta)| \leqslant \sup_{|x|\geqslant R-k, \theta\in\overline{I}} |v(x,\theta)| \to 0 \quad \text{as} \quad R \to +\infty,$$

so that Condition 4 is satisfied for $\varphi$.

In addition, the uniqueness of the solutions constrains the relation

$$\text{if} \quad \theta = 0, \qquad x < x' \;\Rightarrow\; \varphi(x,\theta) < \varphi(x',\theta)$$

to extend to any value of $\theta$, so that the map $x \mapsto \varphi(x,\theta)$ is increasing. Now, suppose that the value $x_0$ is not attained by the map $x \mapsto \varphi(x,\theta_0)$ for a given value $\theta_0$. By considering the ODE 11.13 submitted to initial condition $X(\theta_0) = x_0$, we obtain the existence of a value $X(0)$ such that $\varphi(X(0), \theta_0) = x_0$, which is a contradiction. Hence, the map $x \mapsto \varphi(x,\theta_0)$ is surjective and Condition 1 is satisfied. If $I = S^1$, Equation 11.11 is satisfied by $\varphi$ simply because $v$ is $2\pi$-periodic in the $\theta$ variable.

Last, a classical theorem (dependency with initial conditions, see [7] for example) states that $\varphi$ is $C^n$ and we can write

$$\frac{d}{d\theta}\left(u(\varphi(x,\theta),\theta)\right) = \varphi_\theta(x,\theta)u_x(\varphi(x,\theta),\theta) + u_\theta(\varphi(x,\theta),\theta) = (vu_x + u_\theta)\left(\varphi(x,\theta),\theta\right) = 0. \tag{11.14}$$

Then, integrating Equation 11.14 yields for any $(x,\theta) \in \mathbb{R} \times \overline{I}$,

$$u(\varphi(x,\theta),\theta) = u(\varphi(x,0),0) = u(x,0),$$

so that Condition 2 is satisfied and $\varphi$ is a $\theta$-graph of $u$ of class $C^n$. □

**Lemma 22** *Let $u_0 \in \mathcal{V}_c^n$ $(n \geqslant 2)$, and $\varphi_0$ be a $\theta$-graph of $u_0$ of class $C^n$. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation*

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial \theta^2} \tag{11.15}$$

*on $\Omega^\star = \mathbb{R} \times I^\star \times ]0,+\infty[$ submitted to the boundary condition*

$$\forall (x,\theta,t) \in \partial\Omega^\star, \ \ \varphi(x,\theta,t) = \varphi_0(x,\theta). \tag{11.16}$$

*Then, the unique map $u : \overline{\Omega} \to \mathbb{R}$ defined by*

$$\forall (x,\theta,t) \in \overline{\Omega}, \ \ u(\varphi(x,\theta,t),\theta,t) = u_0(x,0) \tag{11.17}$$

*belongs to $\mathcal{V}_c^{n,n}$ and is a classical solution of the DCMA associated to the initial datum $u_0$.*

***Proof :***

1. Since the heat equation satisfies the comparison principle, the condition

$$x < x' \ \Rightarrow \ \varphi_0(x,\cdot) < \varphi_0(x',\cdot)$$

is preserved along evolution so that

$$x < x' \ \Rightarrow \ \forall \theta, t, \quad \varphi(x,\theta,t) < \varphi(x',\theta,t).$$

and $x \mapsto \varphi(x,\theta,t)$ is increasing as expected.

2. Now we prove that $x \mapsto \varphi(x,\theta,t)$ is surjective. Condition 4 of Definition 24 shows that we can find two constants $A$ and $B$ (with $B = 0$ if $I^\star$ is bounded) such that

$$|\varphi_0(x,\theta) - x| \leqslant A + B|\theta|$$

on $\mathbb{R} \times \overline{I}$. If $I = S^1$, Equation 11.11 extends this property to $\mathbb{R} \times \overline{I^\star}$. A simple result about the heat Equation (see appendix to follow) states that

$$\forall (x,\theta,t) \in \overline{\Omega^\star}, \quad |\varphi(x,\theta,t) - x| \leqslant A + B|\theta| + B\sqrt{\frac{4t}{\pi}}. \tag{11.18}$$

As a consequence, for any $(\theta,t) \in \overline{I} \times ]0,+\infty[$, $x \mapsto \varphi(x,\theta,t)$ is surjective.

3. Hence, Equation 11.17 defines a unique map $u : \overline{\Omega} \to \mathbb{R}$ and a proof similar to the one of Lemma 21 shows that $u \in \mathcal{V}_c^{n,n}$ thanks to Equation 11.18.

4. We check the boundary condition. For any $(x,\theta,t)\in\partial\Omega$, due to Equation 11.16 we have

$$\varphi(x,\theta,t)=\varphi_0(x,\theta),$$

while the definition of $u$ (Equation 11.17) implies

$$u(\varphi(x,\theta,t),\theta,t)=u_0(x,0)=u_0(\varphi_0(x,\theta),\theta),$$

and consequently

$$u(\varphi_0(x,\theta),\theta,t)=u_0(\varphi_0(x,\theta),\theta).$$

Hence, the boundary condition (iii) of Definition 22 is satisfied since the map

$$\begin{array}{ccc}\partial\Omega & \to & \partial\Omega\\ (x,\theta,t) & \mapsto & (\varphi_0(x,\theta),\theta,t)\end{array}$$

is bijective.

5. Let us note $\boldsymbol{z}_1=(\varphi(\boldsymbol{z}),\theta,t)$ for a given $\boldsymbol{z}\in\Omega$. If $u_x(\boldsymbol{z}_1)=0$, differentiating Equation 11.17 with respect to $t$ yields

$$\varphi_t(\boldsymbol{z})u_x(\boldsymbol{z}_1)+u_t(\boldsymbol{z}_1)=u_t(\boldsymbol{z}_1)=0$$

as expected. If $u_x(\boldsymbol{z}_1)\neq 0$, we obtain

$$u_t(\boldsymbol{z}_1)=-\varphi_t(\boldsymbol{z})u_x(\boldsymbol{z}_1),$$

$$\text{and}\qquad \frac{d}{d\theta}(u_0(x,0))=0=\varphi_\theta(\boldsymbol{z})u_x(\varphi(\boldsymbol{z}),\theta,t)+u_\theta(\varphi(\boldsymbol{z}),\theta,t),$$

as well as

$$\begin{aligned}0 &= \frac{d^2}{d\theta^2}(u_0(x,0))\\ &= \frac{d}{d\theta}\left(\varphi_\theta(\boldsymbol{z})u_x(\varphi(\boldsymbol{z}),\theta,t)+u_\theta(\varphi(\boldsymbol{z}),\theta,t)\right)\\ &= \varphi_{\theta\theta}(\boldsymbol{z})u_x(\varphi(\boldsymbol{z}),\theta,t)+\varphi_\theta^2(\boldsymbol{z})u_{xx}(\varphi(\boldsymbol{z}),\theta,t)+2\varphi_\theta(\boldsymbol{z})u_{x\theta}(\varphi(\boldsymbol{z}),\theta,t)+u_{\theta\theta}(\varphi(\boldsymbol{z}),\theta,t)\\ &= \varphi_t(\boldsymbol{z})u_x(\boldsymbol{z}_1)+\varphi_\theta^2(\boldsymbol{z})u_{xx}(\boldsymbol{z}_1)+2\varphi_\theta(\boldsymbol{z})u_{x\theta}(\boldsymbol{z}_1)+u_{\theta\theta}(\boldsymbol{z}_1)\\ &= \left(-u_t+u_{\theta\theta}-2\frac{u_\theta}{u_x}u_{\theta x}+(\frac{u_\theta}{u_x})^2u_{xx}\right)(\boldsymbol{z}_1),\end{aligned}$$

so that condition (ii) of Definition 22 is satisfied. Hence, $u$ is a classical solution of the DCMA associated to the initial datum $u_0$. □

Lemma 22 proves that the DCMA Equation is a scalar formulation of the monodimensional heat equation (11.15), like two other important equations of image processing : the Mean Curvature Motion and the Affine Morphological Scale Space. The difference between them only comes from the intrinsic parameter of the level lines : the Euclidean abscissa for the Mean Curvature Motion, the affine abscissa for the Affine Scale space. For the DCMA, the natural parameter is the time $\theta$, which means that level lines are not considered as curves but as **graphs**. This remark will permit to prove the existence of weak solutions for the DCMA, but in certain cases only, namely, when the level lines of the initial datum can be described by graphs.

## 11.2 Weak solutions of the DCMA

We define weak (only continuous) solutions of the DCMA as uniform limits of classical solutions.

**Definition 25** *Given a movie $u_0 \in \mathcal{C}_c^0$, we say that a map $u \in \mathcal{C}_c^{0,0}$ is a weak solution of the DCMA associated to the initial datum $u_0$ if*

$$\forall (x,\theta,t) \in \partial\Omega, \quad u(x,\theta,t) = u_0(x,\theta)$$

*and if there exists a sequence $(u^\varepsilon)_{\varepsilon>0}$ of classical solutions of the DCMA such that $u^\varepsilon \to u$ uniformly on $\overline{\Omega}$ when $\varepsilon \to 0$.*

**Lemma 23 (uniqueness)** *A weak solution of the DCMA associated to a given initial datum is unique.*

***Proof :***

We simply prove that the contraction property (Corollary 9) is still satisfied. Let $u$ and $\tilde{u}$ be two weak solutions of the DCMA associated to the initial data $u_0$ and $\tilde{u}_0$. Then, we can find two sequences $u^\varepsilon$ and $\tilde{u}^\varepsilon$ which converge uniformly towards $u$ and $\tilde{u}$. Writing $u_0^\varepsilon = u^\varepsilon(\cdot,\cdot,0)$ and $\tilde{u}_0^\varepsilon = \tilde{u}^\varepsilon(\cdot,\cdot,0)$, Corollary 9 ensures that

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_\infty \leqslant \|u_0^\varepsilon - \tilde{u}_0^\varepsilon\|,$$

and taking the (uniform) limits when $\varepsilon \to 0$ yields

$$\|u - \tilde{u}\|_\infty \leqslant \|u_0 - \tilde{u}_0\|$$

as expected. □

**Proposition 28 (existence)** *Call $\overline{\mathcal{V}_c}$ the topological closure of $\mathcal{V}_c^2$ with respect to the $\|\cdot\|_\infty$ norm. Then, given $u_0 \in \overline{\mathcal{V}_c}$, there exists a unique weak solution $u$ of the DCMA associated to the initial datum $u_0$.*

***Proof :***

According to the hypothesis on $u_0$, we can find a sequence $u_0^\varepsilon \in \mathcal{V}_c^2$ which converges uniformly towards $u_0$. Then, call $u^\varepsilon$ the classical solution of the DCMA associated to the initial datum $u_0^\varepsilon$ (Proposition 27 ensures the existence of $u^\varepsilon$). Lemma 20 forces $u^\varepsilon$ to converge uniformly towards a limit $u \in \mathcal{C}_c^{0,0}$, which is by construction a weak solution of the DCMA. □

To make more precise this existence property, we now build explicit weak solutions. The construction is similar to the one used for classical solutions in the proof of Lemma 22.

**Definition 26** *We write $\mathcal{V}_c^0$ the space of movies $u \in \mathcal{C}_c^0$ which admit a continuous $\theta$-graph.*

This generalizes Definition 23 thanks to Lemma 21.

**Proposition 29** *Let $u_0 \in \mathcal{V}_c^0$, and $\varphi_0$ be a $\theta$-graph of $u_0$. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation 11.15 submitted to the boundary condition 11.16. Then, the unique map $u$ defined from $\varphi$ by Equation 11.17 is a weak solution of the DCMA.*

***Proof :***

1. As for the definition of $u$ and its belonging to $\mathcal{C}_c^{0,0}$, the proof is already contained in Lemma 22.

2. Since $\mathcal{V}_c^0 \subset \overline{\mathcal{V}_c}$, we can consider $\tilde{u}$ the weak solution of the DCMA associated to the initial datum $u_0$, and $(u^\varepsilon)$ a sequence of classical solutions which converges uniformly towards $\tilde{u}$. Now we want to prove that $u = \tilde{u}$, or, equivalently, that

$$\forall (x,\theta,t) \in \overline{\Omega}, \quad \tilde{u}(\varphi(x,\theta,t),\theta,t) = u_0(x,0).$$

Given $x_0 \in \mathbb{R}$, $\varepsilon > 0$, $\alpha > 0$ and $T > 0$, define

$$\Lambda(\theta,t) = u^\varepsilon(\varphi(x_0,\theta,t),\theta,t) - u_0(x_0,0) - \alpha t.$$

Since $\Lambda$ is continuous on the compact set $K_T = \overline{I} \times [0,T]$, there exists $(\theta_0,t_0) \in K_T$ such that

$$\max_{K_T} \Lambda = \Lambda(\theta_0,t_0).$$

2.a. Suppose that

$$\theta_0 \in I \quad \text{and} \quad t_0 > 0. \tag{11.19}$$

Then, in $(\theta_0,t_0)$ we have

$$\Lambda_t \geqslant 0, \quad \Lambda_\theta = 0 \quad \text{and} \quad \Lambda_{\theta\theta} \leqslant 0.$$

This yields

$$\varphi_t u_x^\varepsilon + u_t^\varepsilon \geqslant \alpha, \tag{11.20}$$

$$\varphi_\theta u_x^\varepsilon + u_\theta^\varepsilon = 0, \tag{11.21}$$

$$\text{and} \quad \varphi_\theta^2 u_{xx}^\varepsilon + 2\varphi_\theta u_{\theta x}^\varepsilon + u_{\theta\theta}^\varepsilon + \varphi_{\theta\theta} u_x^\varepsilon \leqslant 0. \tag{11.22}$$

If $u_x^\varepsilon = 0$, then $u_t^\varepsilon = 0$, which is in contradiction with Equation 11.20. If $u_x^\varepsilon \neq 0$, since $\varphi_t = \varphi_{\theta\theta}$ and $u^\varepsilon$ is a classical solution of the DCMA, Equation 11.21 and 11.22 imply

$$u_t^\varepsilon + \varphi_t u_x^\varepsilon \leqslant 0,$$

which contradicts Equation 11.20 as well.

2.b. Hence, Assumption 11.19 is false and we have either $\theta_0 \in \partial I$ or $t_0 = 0$, so that $\varphi(x_0, \theta_0, t_0) = \varphi_0(x_0, \theta_0)$. Writing $u_0^\varepsilon = u^\varepsilon(\cdot, \cdot, 0)$, we get

$$\begin{aligned} u^\varepsilon(\varphi(x_0, \theta_0, t_0), \theta_0, t_0) &= u_0^\varepsilon(\varphi_0(x_0, \theta_0), \theta_0) \\ &\leqslant u_0(\varphi_0(x_0, \theta_0), \theta_0) + \|u_0^\varepsilon - u_0\|_\infty \\ &\leqslant u_0(x_0, 0) + \|u_0^\varepsilon - u_0\|_\infty, \end{aligned}$$

so that

$$\forall (x, \theta, t) \in \mathbb{R} \times K_T, \qquad u^\varepsilon(\varphi(x, \theta, t), \theta, t) \leqslant u_0(x, 0) + \alpha T + \|u_0^\varepsilon - u_0\|_\infty.$$

Then, sending $\alpha$ to zero and $T$ to infinity yields

$$\forall (x, \theta, t) \in \overline{\Omega}, \qquad u^\varepsilon(\varphi(x, \theta, t), \theta, t) \leqslant u_0(x, 0) + \|u_0^\varepsilon - u_0\|_\infty,$$

and passing to the limit when $\varepsilon \to 0$ establishes

$$\forall (x, \theta, t) \in \overline{\Omega}, \qquad \tilde{u}(\varphi(x, \theta, t), \theta, t) \leqslant u_0(x, 0).$$

A symmetrical reasoning proves that $\tilde{u}(\varphi(x, \theta, t), \theta, t) \geqslant u_0(x, 0)$ as well, so that $u = \tilde{u}$ as announced. □

A consequence of this characterization of weak solutions is that a weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^n$ admits a kind of velocity movie as soon as $u_0$ is locally Lipschitz in the $x$ variable. To simplify the proof, we directly assume that the whole analysis $u$ is locally Lipschitz in the $x$ variable, although it is not difficult to see that $u$ inherits this property from the initial datum $u_0$.

**Corollary 11** *Let $u$ be the weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^0$. If $u$ is locally Lipschitz in the $x$ variable, then there exists a continuous map $v$ defined on $\Omega = \mathbb{R} \times I \times ]0, +\infty[$ such that on $\Omega$,*

$$u(x + \tau v(x, \theta, t), \theta + \tau, t) = u(x, \theta, t) + o(\tau) \tag{11.23}$$

$$\text{and} \quad u(x + \tau v(x, \theta, t), \theta + \tau, t - \frac{\tau^2}{2}) = u(x, \theta, t) + o(\tau^2). \tag{11.24}$$

***Proof :***

We associate $\varphi$ to $u_0$ as in Proposition 29, and define $v$ by Equation 11.12. Then,

$$\begin{aligned} u_0(x, 0) &= u\left(\varphi(x, \theta + \tau, t), \theta + \tau, t\right) \\ &= u\left(\varphi(x, \theta, t) + \tau \varphi_\theta(x, \theta, t) + o(\tau), \theta + \tau, t\right) \\ &= u\left(\varphi(x, \theta, t) + \tau v(\varphi(x, \theta, t), \theta, t), \theta + \tau, t\right) + o(\tau), \end{aligned}$$

which establishes the first equality. For the second one, we write

$$\begin{aligned} u_0(x,0) &= u\left(\varphi(x,\theta+\tau,t-\frac{\tau^2}{2}),\theta+\tau,t-\frac{\tau^2}{2}\right) \\ &= u\left(\varphi(x,\theta,t)+\tau\varphi_\theta(x,\theta,t)+o(\tau^2),\theta+\tau,t-\frac{\tau^2}{2}\right) \\ &= u\left(\varphi(x,\theta,t)+\tau v(\varphi(x,\theta,t),\theta,t),\theta+\tau,t-\frac{\tau^2}{2}\right)+o(\tau^2) \end{aligned}$$

and the proof is complete. □

**Remark :** Defining the Lie derivative of a map $f$ along the vector $\xi = (v, 1)$ by

$$f_\xi(x,\theta,t) = \left(\frac{d}{d\tau} f(x+\tau v(x,\theta,t),\theta+\tau,t)\right)_{\tau=0},$$

Equation 11.24 is equivalent to $u_\xi = 0$. As concerns Equation 11.24, it implies

$$\left(\frac{d^{[2]}}{d\tau^2} u(x+\tau v(x,\theta,t),\theta+\tau,t-\frac{\tau^2}{2})\right)_{\tau=0} = 0,$$

where the notation $d^{[2]}f/d\tau^2$ means the pseudo-second derivative of $f$, defined in $x$ by

$$\frac{d^{[2]}f}{dx^2}(x) = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}.$$

Notice that this property is a generalization of Equation 11.9.

## 11.3 A viscosity formulation

We now establish the link between our definition of weak solutions and the theory of viscosity solutions (see [27] for further details on viscosity solutions). For the DCMA, defining viscosity solutions is not necessary because smooth movies remain smooth, which permits the previous construction of weak solutions as uniform limits of smooth solutions. However, this is not generally the case with non-linear parabolic PDE of the kind

$$\frac{\partial u}{\partial t} = F(D^2u, Du, u)$$

defined from an elliptic operator $F$ (consider the Mean Curvature Motion or the Affine Morphological Scale Space for example). Moreover, it is convenient to define weak solutions intrinsically, without using limits of regular solutions. In the following, we give a reasonable definition of a viscosity solution of the DCMA, and prove that a weak solution is a viscosity solution.

**Definition 27** *A bounded continuous map $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity subsolution of the DCMA if for any $\phi \in C^\infty(\overline{\Omega})$, at any point $\mathbf{z}_0 \in \Omega$ where $u - \phi$ attains a local maximum, we have* *(i) If* $\phi_x \neq 0$*, then* $\phi_t \leqslant \phi_{\theta\theta} - 2\frac{\phi_\theta}{\phi_x}\phi_{\theta x} + (\frac{\phi_\theta}{\phi_x})^2\phi_{xx}$,

*(ii) If* $\phi_x = 0$*, then* $\phi_\theta = \phi_t = 0$ *and* $\exists\lambda \in I\!R$ *such that* $0 \leqslant \phi_{\theta\theta} + 2\lambda\phi_{\theta x} + \lambda^2\phi_{xx}$.

Condition (i) is the classical formulation of viscosity subsolutions, whereas (ii) is a degenerate condition particular to the DCMA (see [31], [9] for examples of degenerate viscosity solutions in the case of the Mean Curvature Motion).

The definition of a supersolution is symmetrical :

**Definition 28** *A bounded continuous map* $u : \overline{\Omega} \to I\!R$ *is a viscosity supersolution of the DCMA if for any* $\phi \in C^\infty(\overline{\Omega})$*, at any point* $\boldsymbol{z}_0 \in \Omega$ *where* $u - \phi$ *attains a local minimum, we have*

*(i') If* $\phi_x \neq 0$*, then* $\phi_t \geqslant \phi_{\theta\theta} - 2\frac{\phi_\theta}{\phi_x}\phi_{\theta x} + (\frac{\phi_\theta}{\phi_x})^2\phi_{xx}$,

*(ii') If* $\phi_x = 0$*, then* $\phi_\theta = \phi_t = 0$ *and* $\exists\lambda \in I\!R$ *such that* $0 \geqslant \phi_{\theta\theta} + 2\lambda\phi_{\theta x} + \lambda^2\phi_{xx}$.

We give the following equivalent definition of a subsolution for completeness.

**Proposition 30** *A bounded continuous map* $u : \overline{\Omega} \to I\!R$ *is a viscosity subsolution of the DCMA if for any* $(\boldsymbol{p}, A) \in I\!R^3 \times S_3$ *and* $\boldsymbol{z}_0 \in \Omega$ *such that*

$$u(\boldsymbol{z}) \leqslant u(\boldsymbol{z}_0) + \boldsymbol{p}\cdot(\boldsymbol{z}_0 - \boldsymbol{z}) + [A](\boldsymbol{z}_0 - \boldsymbol{z}, \boldsymbol{z}_0 - \boldsymbol{z}) + o(|\boldsymbol{z}_0 - \boldsymbol{z}|^2) \quad \text{as} \quad \boldsymbol{z} \to \boldsymbol{z}_0,$$

*we have, writing* $\boldsymbol{p} = (p_i)$ *and* $A = [a_{ij}]$,

*(i) If* $p_1 \neq 0$*, then* $p_3 \leqslant a_{22} - 2\frac{p_2}{p_1}a_{21} + (\frac{p_2}{p_1})^2 a_{11}$,

*(ii) If* $p_1 = 0$*, then* $p_2 = p_3 = 0$ *and* $\exists\lambda \in I\!R$ *such that* $0 \leqslant a_{22} + 2\lambda a_{21} + \lambda^2 a_{11}$.

The equivalent definition for supersolutions is straightforward.

**Definition 29** *A bounded continuous map* $u : \overline{\Omega} \to I\!R$ *is a viscosity solution of the DCMA if it is both a viscosity super-solution and a viscosity sub-solution.*

**Proposition 31** *Given an initial datum* $u_0 \in \mathcal{V}_c^0$*, the unique weak solution of the DCMA is a viscosity solution.*

***Proof :***

Let $u$ be the weak solution of the DCMA associated to the initial datum $u_0$. We prove that $u$ is a viscosity subsolution of the DCMA. Consider $\phi \in C^\infty(\overline{\Omega})$, and suppose that $u - \phi$ attains a local maximum in $\boldsymbol{z}_0 = (x_0, \theta_0, t_0) \in \Omega$. Let $\varphi$ be the map defined from $u_0$ as in Proposition

29, and define $\boldsymbol{z}_1 = (x_1, \theta_0, t_0)$ by $\varphi(\boldsymbol{z}_1) = x_0$. Then, for $b$ and $c$ in a vicinity of 0 (actually such that $\theta_0 + b \in I$ and $t_0 + c > 0$),

$$u(\varphi(x_1, \theta_0 + b, t_0 + c), \theta_0 + b, t_0 + c) = u_0(x_1, 0) = u(\varphi(x_1, \theta_0, t_0), \theta_0, t_0) = u(\boldsymbol{z}_0).$$

We can estimate

$$\begin{aligned} a(b,c) &:= \varphi(x_1, \theta_0 + b, t_0 + c) - x_0 \\ &= \varphi(x_1, \theta_0 + b, t_0 + c) - \varphi(x_1, \theta_0, t_0) \\ &= b\varphi_\theta(\boldsymbol{z}_1) + \frac{b^2}{2}\varphi_{\theta\theta}(\boldsymbol{z}_1) + c\varphi_t(\boldsymbol{z}_1) + o(b^2 + c) \end{aligned}$$

as $b, c \to 0$. Now, since $u - \phi$ attains a local maximum in $\boldsymbol{z}_0$ and

$$u(x_0 + a(b,c), \theta_0 + b, t_0 + c) - u(x_0, \theta_0, t_0) = 0,$$

we have

$$\begin{aligned} 0 &\leqslant \phi(x_0 + a(b,c), \theta_0 + b, t_0 + c) - \phi(x_0, \theta_0, t_0) \\ &\leqslant a(b,c)\phi_x(\boldsymbol{z}_0) + b\phi_\theta(\boldsymbol{z}_0) + c\phi_t(\boldsymbol{z}_0) + \frac{a^2(b,c)}{2}\phi_{xx}(\boldsymbol{z}_0) + ba(b,c)\phi_\theta(\boldsymbol{z}_0) + \frac{b^2}{2}\phi_{\theta\theta}(\boldsymbol{z}_0) + o(b^2 + c) \\ &\leqslant \left(b\varphi_\theta(\boldsymbol{z}_1) + \frac{b^2}{2}\varphi_{\theta\theta}(\boldsymbol{z}_1) + c\varphi_t(\boldsymbol{z}_1)\right)\phi_x(\boldsymbol{z}_0) + b\phi_\theta(\boldsymbol{z}_0) + c\phi_t(\boldsymbol{z}_0) + \frac{b^2}{2}\varphi_\theta^2(\boldsymbol{z}_1)\phi_{xx}(\boldsymbol{z}_0) \\ &\quad + b^2\varphi_\theta(\boldsymbol{z}_1)\phi_{\theta x}(\boldsymbol{z}_0) + \frac{b^2}{2}\phi_{\theta\theta}(\boldsymbol{z}_0) + o(b^2 + c). \end{aligned}$$

Necessarily, both factors of $b$ and $c$ must be zero and the factor of $b^2$ must be nonnegative. This yields

$$\varphi_\theta(\boldsymbol{z}_1)\phi_x(\boldsymbol{z}_0) + \phi_\theta(\boldsymbol{z}_0) = 0, \tag{11.25}$$

$$\varphi_t(\boldsymbol{z}_1)\phi_x(\boldsymbol{z}_0) + \phi_t(\boldsymbol{z}_0) = 0, \tag{11.26}$$

$$\text{and}\quad \varphi_{\theta\theta}(\boldsymbol{z}_1)\phi_x(\boldsymbol{z}_0) + \varphi_\theta^2(\boldsymbol{z}_1)\phi_{xx}(\boldsymbol{z}_0) + 2\varphi_\theta(\boldsymbol{z}_1)\phi_{\theta x}(\boldsymbol{z}_0) + \phi_{\theta\theta}(\boldsymbol{z}_0) \geqslant 0, \tag{11.27}$$

but as $\varphi_t(\boldsymbol{z}_1) = \varphi_{\theta\theta}(\boldsymbol{z}_1)$, Equation 11.26 and 11.27 imply

$$\phi_t(\boldsymbol{z}_0) \leqslant \varphi_\theta^2(\boldsymbol{z}_1)\phi_{xx}(\boldsymbol{z}_0) + 2\varphi_\theta(\boldsymbol{z}_1)\phi_{\theta x}(\boldsymbol{z}_0) + \phi_{\theta\theta}(\boldsymbol{z}_0). \tag{11.28}$$

1. If $\phi_x(\boldsymbol{z}_0) \neq 0$, Equation 11.25 gives

$$\varphi_\theta(\boldsymbol{z}_1) = -\frac{\phi_\theta}{\phi_x}(\boldsymbol{z}_0)$$

and Equation 11.28 leads to the desired condition (i).

2. If $\phi_x(\boldsymbol{z}_0) = 0$, then $\phi_\theta = \phi_t(\boldsymbol{z}_0) = 0$ is a consequence of Equation 11.25 and 11.26, while Equation 11.28 implies that the polynomial

$$X \mapsto X^2\phi_{xx}(\boldsymbol{z}_0) + 2X\phi_{\theta x}(\boldsymbol{z}_0) + \phi_{\theta\theta}(\boldsymbol{z}_0)$$

takes at least one nonnegative value, which is the desired condition (ii).

Consequently, $u$ is a viscosity subsolution of the DCMA. A symmetrical reasoning shows that it is a viscosity supersolution as well. □

We conjecture that a viscosity solution associated to a given initial datum is unique. In particular, this would imply that the viscosity and the weak solutions of the DCMA are the same, provided that the initial datum $u_0$ lies in $\mathcal{V}_c^0$.

## 11.4 Appendix on the heat equation

In the previous section, we used several results about the monodimensional heat equation. For completeness, we briefly recall them. In all the following, either $J = ]\theta_1, \theta_2[$ or $J = \mathbb{R}$, and $\Omega = J \times ]0, +\infty[$.

**Proposition 32** *Given a continuous map $f : \overline{J} \to \mathbb{R}$ such that*

$$\exists A, B,\ \forall \theta \in \overline{J}, \quad |f(\theta)| \leqslant A + B|\theta|, \tag{11.29}$$

*there exists a unique continuous map $\varphi : \overline{\Omega} \to \mathbb{R}$ such that*

*(i) on $\Omega$, $(\theta, t) \mapsto \varphi(\theta, t)$ is $C^2$ with respect to $\theta$ and $C^1$ with respect to $t$,*

*(ii) on $\Omega$, $\dfrac{\partial \varphi}{\partial t} = \dfrac{\partial^2 \varphi}{\partial \theta^2}$,*

*(iii) for any $(\theta, t) \in \partial\Omega$, $\varphi(\theta, t) = f(\theta)$*

*(iv) $\forall T > 0$, $\exists A, B$, $\forall (\theta, t) \in \overline{J} \times [0, T]$, $\quad |\varphi(\theta, t)| \leqslant A + B|\theta|$.*

**Remark :** If $J$ is bounded, then Equation 11.29 simply means that $f$ is bounded, and Condition (iv) means that $\varphi$ is bounded too. If $J = \mathbb{R}$, $f$ and $\varphi$ are constrained to be "sub-linear" in the $\theta$ variable.

We give a quick justification of Proposition 32 since the heat equation is generally considered for bounded maps in the literature. (see [16] for example). As for the uniqueness, it results from the following comparison principle.

**Proposition 33 (comparison principle)** *Consider $\varphi$ a solution of the heat equation (in the sense of Proposition 32) associated to the initial datum $f \leqslant 0$. Then, $\varphi \leqslant 0$.*

***Proof :***

Suppose first that $J = \mathbb{R}$. Given $T > 0$, there exists $A, B$ such that

$$\forall (\theta, t) \in \overline{J} \times [0, T], \quad \varphi(\theta, t) - A - B|\theta| \leqslant 0. \tag{11.30}$$

For $R > 0$ fixed, we consider the map

$$\Lambda(\theta, t) = \varphi(\theta, t) - (\frac{A}{R^2} + \frac{B}{R})(\theta^2 + 2t).$$

$\Lambda$ satisfies the heat equation, and the maximum principle (see [16] for example) tells that

$$\max_{[-R,R]\times[0,T]} \Lambda = \max_{[-R,R]\times\{0\}\cup\{-R,R\}\times[0,T]} \Lambda.$$

On $[-R, R] \times \{0\}$, $\Lambda \leqslant 0$ because $f \leqslant 0$, while Equation 11.30 yields $\Lambda \leqslant 0$ on $\{-R, R\} \times [0, T]$. Hence, we have

$$\forall(\theta, t) \in [-R, R] \times [0, T], \quad \varphi(\theta, t) \leqslant (\frac{A}{R^2} + \frac{B}{R})(\theta^2 + 2t).$$

Sending $R$ to infinity yields

$$\forall(\theta, t) \in \overline{J} \times [0, T], \quad \varphi(\theta, t) \leqslant 0, \tag{11.31}$$

so that $\varphi \leqslant 0$ on $\overline{J} \times [0, +\infty[$. If $J$ is bounded, Equation 11.31 is a direct consequence of the maximum principle applied to $\varphi$, and the conclusion still holds. □

Now we give an explicit construction of solutions. If $J = \mathbb{R}$, the solution is given by the convolution with the Gaussian kernel :

$$\varphi(\theta, t) = \int_{-\infty}^{+\infty} f(\theta - \alpha) \frac{1}{\sqrt{4\pi t}} e^{-\alpha^2/4t} d\alpha.$$

If $J = ]\theta_1, \theta_2[$, we write $\tilde{f}(\theta) = f(\theta) - l(\theta)$, where $l$ is the unique affine map which forces $\tilde{f}(\theta_1) = \tilde{f}(\theta_2) = 0$. Then, we extend $\tilde{f}$ to an odd and $2(\theta_2 - \theta_1)$-periodic map and apply the previous convolution formula. This way, we obtain a map $\tilde{\varphi}$ which satisfies conditions (i), (ii) and (iv) as well as $\tilde{\varphi}(\theta_1, t) = \tilde{\varphi}(\theta_2, t) = 0$ for any $t \geqslant 0$. Last, the map

$$\begin{aligned} \varphi \quad : \quad & \overline{J} \times [0, +\infty[ \to \mathbb{R} \\ & (\theta, t) \mapsto \tilde{\varphi}(\theta, t) + l(\theta) \end{aligned}$$

satisfies the desired conditions (i), (ii), (iii) and (iv).

**Proposition 34** *Here we suppose $J = \mathbb{R}$. If $f$ satisfies*

$$\forall\theta \in \mathbb{R}, \quad |f(\theta)| \leqslant A + B|\theta|,$$

*then the solution $\varphi$ of the heat equation with initial datum $f$ satisfies*

$$\forall(\theta, t) \in \mathbb{R} \times [0, +\infty[, \quad |\varphi(\theta, t)| \leqslant A + B|\theta| + B\sqrt{\frac{4t}{\pi}}.$$

***Proof :***

Calling $G_t$ the Gaussian kernel, we have

$$|\varphi(\theta,t)| \leqslant \int_{-\infty}^{+\infty} |f(\theta-\alpha)|G_t(\alpha)d\alpha \leqslant \int_{-\infty}^{+\infty} (A+B|\theta|+B|\alpha|)G_t(\alpha)d\alpha,$$

and the announced result is a consequence of the equalities

$$\int_{-\infty}^{+\infty} G_t(\alpha)d\alpha = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} |\alpha|G_t(\alpha)d\alpha = \sqrt{\frac{4t}{\pi}}.$$

□

## 11.5 Further existence properties

In the previous sections, we did not prove the existence of (weak or classical) solutions of the DCMA in the general case, that is to say when the initial datum admits no $\theta$-graph. In fact, we do not believe that the DCMA admits a solution in general, at least a solution in the sense we defined.

When the initial datum $u_0$ admits a $\theta$-graph, the DCMA is obtained by applying the linear monodimensional heat equation to the level lines of $u_0$. For an ordinary continuous map $u_0$, the level lines have no reason to be graphs in the $\theta$ variable, since to a given value of $\theta$, several values of $x$ will correspond in general. Hence, defining general solutions of the DCMA is somewhat equivalent to defining solutions of the heat equation for multi-valued data. It is in that spirit that L.C.Evans studied independently Equation 10.2 in his article "A geometric Interpretation of the Heat Equation with Multivalued Initial Data" (see [32]). He regards the DCMA Equation as the limit when $\varepsilon \to 0$ of the more regular equation

$$u_t = \frac{u_x^2 u_{\theta\theta} - 2u_x u_\theta u_{x\theta} + u_\theta^2 u_{xx}}{u_x^2 + \varepsilon^2 u_\theta^2}. \tag{11.32}$$

Equation 11.32 admits viscosity solutions because it is more or less the Mean Curvature Motion (actually, the case $\varepsilon = 1$ is exactly the Mean Curvature Motion). He noticed that in the general case (that is, when the level lines of the initial datum are not graphs), the regularizing effects of the heat equation are so strong that the limit of approximate solutions is not continuous at scale $t = 0$, because the level lines are constrained to become graphs instantaneously. It seems difficult to overcome this difficulty unless we allow solutions of the DCMA not to be continuous at scale $t = 0$. In fact, it might be possible to define a kind of projection operator which makes the level lines of a movie unfold and become graphs. We shall come back to this when studying a numerical scheme in Chapter 13.

# Chapter 12

# Properties of the DCMA

In this chapter, we investigate several properties of the DCMA. We first check the ones that are constrained by the axiomatic formulation, and then we prove that the DCMA acts as a strong smoothing process along the movement. We also establish integral estimations and try to associate the DCMA to a variational principle. Coming back to the original context of depth interpretation, we finally highlight geometrical properties and find a new characterization of the DCMA.

## 12.1 Checking the axioms

In order to be sure that our axiomatic formulation is consistent, we have to check that the axioms we introduced are satisfied by the DCMA. As regards the three architectural axioms ([**Recursivity**], [**Local Comparison Principle**] and [**Regularity**]), they are direct consequences of the fact that the DCMA is given by an evolution equation

$$u_t = F(D^2u, Du),$$

where $F$ is an elliptic operator. Now we prove that the DCMA satisfies the [**Strong Morphological Invariance**] property.

**Proposition 35** *Let $u$ be a weak solution of the DCMA and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous map. Then, $g \circ u$ is a weak solution of the DCMA.*

***Proof :***

Notice that this proposition makes sense because if $u \in \mathcal{C}_c^0$, then $g \circ u \in \mathcal{C}_{\tilde{c}}^0$ with

$$\tilde{c} = (g(c^-), g(c^+)).$$

1. First, suppose that $u$ is a classical solution of the DCMA and that $g$ is of class $C^2$. Writing $\tilde{u} = g \circ u$, a simple computation gives

$$\tilde{u}_x = (g' \circ u) \cdot u_x, \quad \tilde{u}_t = (g' \circ u) \cdot u_t,$$

and

$$\begin{aligned}\tilde{u}_{\xi\xi} &= \tilde{u}_{\theta\theta} - \frac{\tilde{u}_\theta}{\tilde{u}_x}\tilde{u}_{\theta x} + (\frac{\tilde{u}_\theta}{\tilde{u}_x})^2\tilde{u}_{xx} \\ &= g''\circ u\cdot(u_\theta^2 - 2u_\theta^2 + u_\theta^2) + g'(u)\left(u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\right) \\ &= g'(u)\cdot u_{\xi\xi}\end{aligned}$$

whenever $\tilde{u}_x \neq 0$. Hence, we have $\tilde{u}_t = 0$ if $\tilde{u}_x = 0$, and $\tilde{u}_t = \tilde{u}_{\xi\xi}$ if $\tilde{u}_x \neq 0$, so that $\tilde{u}$ is a classical solution of the DCMA.

2. Now let us come back to the general case when $g$ is only continuous. Given $\varepsilon > 0$, there exists a map $g^\varepsilon \in C^2(\mathbb{R})$ such that $\|g - g^\varepsilon\|_\infty \leqslant \varepsilon$. Since the set

$$K = [-\|u\|_\infty - \varepsilon, \|u\|_\infty + \varepsilon]$$

is compact, $g$ is uniformly continuous on $K$ thanks to Heine's Theorem : in other words there exists a positive number $\alpha \leqslant \varepsilon$ such that $|g(x) - g(y)| \leqslant \varepsilon$ as soon as $|x - y| \leqslant \alpha$. Besides, we can find a classical solution $u^\varepsilon$ of the DCMA such that $\|u - u^\varepsilon\|_\infty \leqslant \alpha$. Then, we have

$$\|g\circ u - g^\varepsilon\circ u^\varepsilon\|_\infty \leqslant \|g\circ u - g\circ u^\varepsilon\|_\infty + \|g\circ u^\varepsilon - g^\varepsilon\circ u^\varepsilon\|_\infty \leqslant 2\varepsilon,$$

and $g^\varepsilon\circ u^\varepsilon$ is a classical solution of the DCMA. □

As for the **[Transversal Invariance]** property, it is clearly satisfied by the DCMA since the $y$ coordinate does not even appear in its definition.

Now we can check the **[v-Compatibility]** property. Consider a map $h : \mathbb{R}^4 \mapsto \mathbb{R}$ such that

$$\forall u \in \mathcal{M}^1,\ R_h u \in \mathcal{M}^1 \quad\text{and}\quad v[R_h u] = v[u],$$

with $R_h u(x, y, \theta) = h\left(u(x, y, \theta), x, y, \theta\right)$. Choosing $u(x, y, \theta) = \lambda\tanh x$ (tanh meaning the hyperbolic tangent) proves that $h$ is $C^1$. In addition, for any $u \in \mathcal{M}^1$ we must have

$$u_x h_\theta = u_\theta h_x$$

in order that the condition $v[R_h u] = v[u]$ is satisfied. If we now choose $u(x, y, \theta) = \tanh x + b\theta$, we obtain $h_\theta = 0$ with $b = 0$ and then $h_x = 0$ with $b = 1$, so that we finally have

$$h(\lambda, x, y, \theta) = f(\lambda, y).$$

Then, the relation $T_t\circ R_h = R_h\circ T_t$ is a direct consequence of Proposition 35, the $y$ coordinate being fixed. □

The last two axioms, **[Galilean Invariance]** and **[Zoom Invariance]**, are clearly satisfied by the DCMA thanks to Lemma 18 and Lemma 19.

## 12.2 Asymptotics of the DCMA

Given an evolution equation like the DCMA, a natural question arises : is there an asymptotic state ? In other words, we would like to know whether the movie $u(\cdot, t)$ tends towards a limit movie $u_\infty$ as $t \to +\infty$.

**Proposition 36** *If $u \in \mathcal{V}_c^0$ is a weak solution of the DCMA, then the limit*

$$u_\infty = \lim_{t \to +\infty} u(\cdot, t)$$

*exists and satisfies*

- *if $I = S^1$, $u_\infty(x, y, \theta) = u_\infty(x, y, 0)$,*
- *if $I = ]\theta_1, \theta_2[$, there exists $v \in C^0(\mathbb{R}^2)$ such that*

$$u_\infty(x - v(x, y)\theta, y, \theta) = u_\infty(x, y, 0).$$

***Proof :***

We proved in Proposition 29 that $u$ satisfies

$$u(\varphi(x, y, \theta, t), y, \theta, t) = u(x, y, 0, 0).$$

Since $\varphi$ is a solution of the heat equation, there exists two maps $a$ and $b$ such that

$$\varphi(x, y, \theta, t) \to a(x, y)\theta + b(x, y) \quad \text{as} \quad t \to +\infty,$$

and if $I = S^1$ the condition

$$\varphi_\theta(x, y, \theta, 0) \to 0 \quad \text{as} \quad |x| \to +\infty$$

forces $a(x, y) = 0$. □

**Remark :** The stronger condition in the case $I = S^1$ is only a consequence of the space of solutions we choose. The main idea that must emerge from this proposition is that the trajectories of the initial movie eventually become straight lines as $t$ reaches infinity.

## 12.3 Diffusion of the movement

In the following, $v$ is a map of class $C^2$ defined on a subset $\Omega'$ of $\Omega = \mathbb{R} \times I \times ]0, +\infty[$, and on $\Omega'$ $v$ satisfies

$$u_\theta + v u_x = 0. \tag{12.1}$$

This defines on $\Omega'$ the operator

$$\frac{\mathrm{D}}{\mathrm{D}\theta} = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x},$$

as well as the notation

$$f_{\xi\xi} = [D^2 f](\xi,\xi) \quad \text{with} \quad \xi = (v,1).$$

**Proposition 37** *Let $u \in \mathcal{C}_c^{n+3,1}$ be a classical solution of the DCMA, with $n \geqslant 0$. Then the movement derivatives (velocity $v = -\frac{u_\theta}{u_x}$, acceleration ? $= \frac{\mathrm{D}v}{\mathrm{D}\theta}$, $\dots \frac{\mathrm{D}^n v}{\mathrm{D}\theta^n}$, $\dots$) are diffused in the same direction as $u$, that is*

$$\forall k \in \{0,\dots n\}, \qquad \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_t = \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_{\xi\xi} \quad \text{whenever} \quad u_x \neq 0.$$

*In particular, the apparent velocity $v$ follows the polynomial and causal diffusion equation*

$$v_t = v_{\theta\theta} + 2vv_{\theta x} + v^2 v_{xx} \quad \text{whenever} \quad u_x \neq 0.$$

To establish this property, it is interesting to introduce the formalism of the Lie brackets associated to the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial \theta}$, $\frac{\partial}{\partial t}$, which commute together, and to the total derivative $\frac{\mathrm{D}}{\mathrm{D}\theta} = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}$.

We compute

$$[\frac{\partial}{\partial x}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = \frac{\partial}{\partial x}\frac{\mathrm{D}}{\mathrm{D}\theta} - \frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}(\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}) - (\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x})\frac{\partial}{\partial x} = v_x\frac{\partial}{\partial x}.$$

One easily checks as well that

$$[\frac{\partial}{\partial\theta}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = v_\theta\frac{\partial}{\partial x} \quad \text{and} \quad [\frac{\partial}{\partial t}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = v_t\frac{\partial}{\partial x}.$$

This way, we can expand the $f_{\xi\xi} = [D^2 f](\xi,\xi)$ notation into

$$\begin{aligned}
(\quad)_{\xi\xi} &= \frac{\partial^2}{\partial\theta^2} + 2v\frac{\partial^2}{\partial\theta\partial x} + v^2\frac{\partial^2}{\partial x^2} \\
&= (\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x})\frac{\partial}{\partial\theta} + v(\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x})\frac{\partial}{\partial x} \\
&= \frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial\theta} + v\frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial x} \\
&= \frac{\mathrm{D}}{\mathrm{D}\theta}(\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}) - \frac{\mathrm{D}v}{\mathrm{D}\theta}\frac{\partial}{\partial x}
\end{aligned}$$

and finally we get, writing ? $= \frac{\mathrm{D}v}{\mathrm{D}\theta}$,

$$(\quad)_{\xi\xi} = \frac{\mathrm{D}^2}{\mathrm{D}\theta^2} - ?\,\frac{\partial}{\partial x}.$$

In particular, if we write $\psi = \frac{\mathrm{D}?}{\mathrm{D}\theta}$ the total derivative of ?, we have

$$v_{\xi\xi} = \psi - ?\,v_x.$$

**Lemma 24** *Independently of any evolution equation, on $\Omega'$ we have*

$$[\frac{\partial}{\partial t} - (\ )_{\xi\xi}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = (v_t - v_{\xi\xi})\frac{\partial}{\partial x}. \tag{12.2}$$

***Proof :***

We compute the Lie bracket

$$\begin{aligned}
[(\,)_{\xi\xi}, \frac{\mathrm{D}}{\mathrm{D}\theta}] &= [\frac{\mathrm{D}^2}{\mathrm{D}\theta^2} - ?\frac{\partial}{\partial x}, \frac{\mathrm{D}}{\mathrm{D}\theta}] \\
&= [\frac{\mathrm{D}^2}{\mathrm{D}\theta^2}, \frac{\mathrm{D}}{\mathrm{D}\theta}] - [?\frac{\partial}{\partial x}, \frac{\mathrm{D}}{\mathrm{D}\theta}] \\
&= 0 + \frac{\mathrm{D}?}{\mathrm{D}\theta}\frac{\partial}{\partial x} - ?[\frac{\partial}{\partial x}, \frac{\mathrm{D}}{\mathrm{D}\theta}] \\
&= (\psi - ?\,v_x)\frac{\partial}{\partial x} \\
&= v_{\xi\xi}\frac{\partial}{\partial x}.
\end{aligned}$$

Now, by linearity, we get as announced

$$[\frac{\partial}{\partial t} - (\,)_{\xi\xi}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = [\frac{\partial}{\partial t}, \frac{\mathrm{D}}{\mathrm{D}\theta}] - [(\,)_{\xi\xi}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = (v_t - v_{\xi\xi})\frac{\partial}{\partial x}.$$

□

***Proof of Proposition 37 :***

We take $\Omega' = \{\boldsymbol{z} \in \Omega,\ u_x(\boldsymbol{z}) \neq 0\}$, so that $v$ is uniquely defined by Equation 12.1 on $\Omega'$. Applying Equation 12.2 to $u$ yields

$$\left(\frac{\partial}{\partial t} - (\ )_{\xi\xi}\right)\frac{\mathrm{D}u}{\mathrm{D}\theta} + \frac{\mathrm{D}}{\mathrm{D}\theta}(u_t - u_{\xi\xi}) = (v_t - v_{\xi\xi})u_x. \tag{12.3}$$

As $u$ satisfies $\frac{\mathrm{D}u}{\mathrm{D}\theta} = 0$ as well as $u_t = u_{\xi\xi}$ on $\Omega'$ ($u$ is solution of the DCMA), the left term of Equation 12.3 is zero. Hence, on $\Omega'$ we have $v_t = v_{\xi\xi}$ as announced in Proposition 37.

This proves that the right term of Equation 12.2 is zero on $\Omega'$, so that

$$[\frac{\partial}{\partial t} - (\ )_{\xi\xi}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = 0 \quad \text{whenever} \quad u_x \neq 0.$$

Consequently, for any $q : \Omega' \to \mathbb{R}$ of class $C^3$ satisfying

$$q_t = q_{\xi\xi},$$

we have

$$\left(\frac{\mathrm{D}q}{\mathrm{D}\theta}\right)_t = \left(\frac{\mathrm{D}q}{\mathrm{D}\theta}\right)_{\xi\xi} \quad \text{whenever} \quad u_x \neq 0.$$

Thus, a simple induction proves that the diffusion equation $q_t = q_{\xi\xi}$ is satisfied by all successive total derivatives of $v$ of class $C^2$, that is, $\dfrac{\mathrm{D}v}{\mathrm{D}\theta}, \ldots, \dfrac{\mathrm{D}^n v}{\mathrm{D}\theta^n}$. □

Now we would like to generalize Proposition 37 to the whole $\Omega$, i.e. even at points where $u_x$ vanishes.

**Proposition 38** *If $u \in \mathcal{V}_c^{n+3,1}$, then there exists a velocity map $v$ associated to $u$ which satisfies,* ***on the whole*** $\Omega$,

$$\forall k \in \{0, \ldots n\}, \qquad \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_t = \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_{\xi\xi}. \tag{12.4}$$

*Moreover, if $I = ]\theta_1, \theta_2[$, then*

$$\forall (x, i, t) \in \mathbb{R} \times \{1, 2\} \times ]0, +\infty[ \qquad ?\,(x, \theta_i, t) = 0. \tag{12.5}$$

***Proof :***

Define $\varphi$ as in Lemma 22, and consider the velocity map $v$ defined by

$$v(\varphi(x, \theta, t), \theta, t) = \varphi_\theta(x, \theta, t). \tag{12.6}$$

1. We get, writing $\boldsymbol{z}_0 = (x, \theta, t)$ and $\boldsymbol{z}_1 = (\varphi(x, \theta, t), \theta, t)$,

$$\begin{aligned} v_t(\boldsymbol{z}_1) &= \varphi_{\theta t}(\boldsymbol{z}_0) - \varphi_t(\boldsymbol{z}_0) v_x(\boldsymbol{z}_1) \\ &= \varphi_{\theta\theta\theta}(\boldsymbol{z}_0) - \varphi_{\theta\theta}(\boldsymbol{z}_0) v_x(\boldsymbol{z}_1), \end{aligned}$$

while

$$\varphi_{\theta\theta}(\boldsymbol{z}_0) = v_\theta(\boldsymbol{z}_1) + \varphi_\theta(\boldsymbol{z}_0) v_x(\boldsymbol{z}_1)$$

and

$$\varphi_{\theta\theta\theta}(\boldsymbol{z}_0) = v_{\theta\theta}(\boldsymbol{z}_1) + 2\varphi_\theta(\boldsymbol{z}_0) v_{\theta x}(\boldsymbol{z}_1) + \varphi_\theta^2(\boldsymbol{z}_0) v_{xx}(\boldsymbol{z}_1) + \varphi_{\theta\theta}(\boldsymbol{z}_0) v_x(\boldsymbol{z}_1).$$

Hence, we have

$$\begin{aligned} v_t(\boldsymbol{z}_1) &= v_{\theta\theta}(\boldsymbol{z}_1) + 2\varphi_\theta(\boldsymbol{z}_0) v_{\theta x}(\boldsymbol{z}_1) + \varphi_\theta^2(\boldsymbol{z}_0) v_{xx}(\boldsymbol{z}_1) \\ &= \left(v_{\theta\theta} + 2 v v_{\theta x} + v^2 v_{xx}\right)(\boldsymbol{z}_1) \\ &= v_{\xi\xi}(\boldsymbol{z}_1) \end{aligned}$$

as expected. This proves that the right term of Equation 12.2 is identically zero on the whole $\Omega$, so that this diffusion property extends to the successive total derivatives of $v$ as we noticed in the proof of Proposition 37.

2. Differentiating Equation 12.6 with respect with $\theta$, we get

$$?\,(\varphi(x, \theta, t), \theta, t) = \varphi_{\theta\theta}(x, \theta, t),$$

so that for any $(x, i, t)$ in $\mathbb{R} \times \{1, 2\} \times ]0, +\infty[$ we have

$$?\,(\varphi(x, \theta_i, t), \theta_i, t) = \varphi_t(x, \theta_i, t) = \frac{\partial}{\partial t}\varphi(x, \theta_i, t) = \frac{\partial}{\partial t}\varphi(x, \theta_i, 0) = 0.$$

$\square$

**Remark :** If $u \in \mathcal{C}_c^{0,0}$ is a weak solution of the DCMA, locally Lipschitz in the $x$ variable, it is possible to establish an equivalent result in the continuation of Corollary 11, provided that we substitute the total derivative $\frac{\mathrm{D}}{\mathrm{D}\theta}$ by the Lie derivative

$$f_\xi(x,\theta,t) := \left(\frac{d}{d\tau} f(x+\tau v(x,\theta,t), \theta+\tau, t)\right)_{\tau=0}.$$

From Corollary 11 we know that there exists a velocity map $v$ (i.e. such that $u_\xi = 0$), defined on $\Omega$, which also constrains

$$u(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = u(x,\theta,t) + o(\tau^2).$$

Then, it is not difficult to show that

$$v(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = v(x,\theta,t) + \tau v_\xi(x,\theta,t) + o(\tau^2).$$

More generally, the successive Lie derivatives of $v$ along the movement are well defined ($? = v_\xi$, $\psi = ?_\xi$, ..., $v^{[n+1]} = (v^{[n]})_\xi$, ...) and satisfy

$$v^{[n]}(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = v^{[n]}(x,\theta,t) + \tau v^{[n+1]}(x,\theta,t) + o(\tau^2).$$

This highlights an interesting property of the DCMA : the velocity field is smoothed indirectly through the anisotropic diffusion of $u$. Notice that the diffusion Equation

$$v_t = v_{\theta\theta} + 2vv_{\theta x} + v^2 v_{xx}$$

presents no singularity and is of the kind

$$v_t = F(D^2 v, Dv, v),$$

where $F$ is a continuous elliptic operator. This means in particular that the classical theory of viscosity solutions (see [27]) applies. Our study goes a little further as $v$ does not necessarily exist at $t = 0$, but we saw that it can be defined for any $t > 0$ as soon as $u_0 \in \mathcal{V}_c^0$. This is a direct consequence of the strong regularization effects of the heat equation.

As regards boundary conditions for $v$ when $u_0$ is regular enough, we have

$$\forall (x,\theta) \in \mathbb{R} \times \overline{I}, \qquad v(x,\theta,0) = v_0(x,\theta)$$

and if $I = ]\theta_1, \theta_2[$, then

$$\forall (x,i,t) \in \mathbb{R} \times \{1,2\} \times ]0,+\infty[, \qquad (v_\theta + vv_x)(x,\theta_i,t) = 0$$

according to Proposition 38.

## 12.4 A conservation law

### 12.4.1 Compactly supported movies

We would like to consider integrals like

$$\int_I \int_{-\infty}^{+\infty} u(x,\theta)\, dx d\theta.$$

To simplify the results, we are going to work on compactly supported movies, which is not very restrictive physically speaking. We first recall the classical

**Definition 30** *A movie $u : \mathbb{R}\times\overline{I} \to \mathbb{R}$ is **compactly supported** if it is zero outside a compact set of $\mathbb{R}\times\overline{I}$.*

Practically, it is equivalent to say that there exists $R > 0$ such that $u(x,\theta) = 0$ as soon as $|x| \geqslant R$.

**Lemma 25** *A compactly supported movie $u \in \mathcal{V}_0^n$ ($n \geqslant 1$) admits a compactly supported velocity map.*

***Proof :***

Suppose that $u(x,\theta) = 0$ when $|x| \geqslant R$ and let $v$ be a velocity map of $u$. There exists a map $\phi \in C^\infty(\mathbb{R})$ such that $\phi(x) = 0$ if $|x| \geqslant R+1$ and $\phi(x) = 1$ if $|x| \leqslant R$. Thus, the map

$$\tilde{v} : (x,\theta) \mapsto \phi(x)\cdot v(x,\theta)$$

is a velocity map of $u$ because $u_\theta = u_x = 0$ when $|x| > R$. Last, it is clear that $\tilde{v}$, as well as $v$, is bounded and of class $C^{n-1}$. □

**Proposition 39** *Let $u$ be the (weak or classical) solution of the DCMA associated to a compactly supported initial datum $u_0 \in \mathcal{V}_0^n$. Then,*

$$\exists R > 0,\ \ \forall (x,\theta,t) \in \mathbb{R}\times\overline{I}\times[0,+\infty[, \qquad |x| \geqslant R+t \ \Rightarrow\ u(x,\theta,t) = 0. \tag{12.7}$$

*and if $n \geqslant 1$, $u$ admits a velocity map which satisfies the same conclusion.*

***Proof :***

This is a simple consequence of Equation 11.18. Recall that the solution $u$ of the DCMA can be defined by

$$\forall (x,\theta,t) \in \overline{\Omega},\ \ u(\varphi(x,\theta,t),\theta,t) = u_0(x,0),$$

where $\varphi$ satisfies

$$\exists C,\ \forall (x,\theta,t) \in \overline{\Omega}, \quad |\varphi(x,\theta,t)| \geqslant |x| - C - t$$

thanks to Equation 11.18. But since $u_0$ is compactly supported, there exists $R > 0$ such that $u_0(x,\theta) = 0$ as soon as $|x| \geqslant R - C$. Then, we have $u(x,\theta,t) = 0$ as soon as $|x| \geqslant t + R$. □

### 12.4.2 Light Energy conservation

**Proposition 40** *Let $u \in \mathcal{V}_0^{2,1}$ be the classical solution of the DCMA associated to a compactly supported initial datum. Suppose that*

*(a) either $I = S^1$,*

*(b) or $I \neq S^1$ and $\forall (x,i) \in \mathbb{R} \times \{1,2\}, \quad u(x,\theta_i,0) = 0$.*

*Then, the light energy at scale $t$, defined by*

$$I(t) = \frac{1}{2} \iint u^2(x,\theta,t)\, dx d\theta,$$

*is independent of $t$.*

***Proof :***

We take the convention $(\theta_1, \theta_2) = (0, 2\pi)$ if $I = S^1$, and remark that if $I = ]\theta_1, \theta_2[$, then the boundary condition on $u$ implies

$$\forall (x,i,t) \in \mathbb{R} \times \{1,2\} \times [0,+\infty[ \quad u(x,\theta_i,t) = u(x,\theta_i,0) = 0$$

thanks to Condition (b), so that

$$\forall (x,i,t) \in \mathbb{R} \times \{1,2\} \times [0,+\infty[ \quad u_x(x,\theta_i,t) = \frac{\partial}{\partial x} u(x,\theta_i,t) = 0.$$

In the following, $v$ is a velocity map associated to $u$. Since $u(\cdot,\cdot,t)$ is compactly supported thanks to Proposition 39, the integral

$$I(t) = \frac{1}{2} \iint u^2 dx d\theta$$

is taken on a compact set. Consequently, as $u \in \mathcal{C}_0^{2,1}$, $I$ is derivable and we can derive under the integral symbol to obtain

$$\begin{aligned} I'(t) &= \iint u u_t\, dx d\theta \\ &= \iint u u_{\xi\xi}\, dx d\theta \\ &= -\iint u u_x ?\ dx d\theta \\ &= -\iint u u_x (v_\theta + v v_x)\, dx d\theta \\ &= -\iint u u_x v_\theta - u u_\theta v_x\, dx d\theta. \end{aligned}$$

By integrating by parts, we get

$$I'(t) = -\int [u u_x v]_{\theta_1}^{\theta_2}\, dx + \int [u u_\theta v]_{-\infty}^{+\infty}\, d\theta + \iint (u u_x)_\theta v - (u u_\theta)_x v\, dx d\theta.$$

The first term is zero thanks to (a) or (b), the second one is zero because $u(\cdot,\cdot,t)$ is compactly supported and $v$ is bounded, and the third one is evidently zero. Hence, $I(t)$ does not depend on $t$. □

## 12.5 A variational principle

### 12.5.1 A minimization law

**Proposition 41** *Let $u \in \mathcal{V}_0^{4,1}$ be the classical solution of the DCMA associated to a compactly supported initial datum. Then the quantity*

$$E(t) = \frac{1}{2} \iint ?^2(x, \theta, t)\, dx d\theta$$

*decreases with scale and we have*

$$\frac{dE}{dt}(t) = - \iint (\frac{\mathrm{D}?}{\mathrm{D}\theta})^2\, dx d\theta. \tag{12.8}$$

***Proof :***

In all the following, $v$ is a velocity field of $u$ satisfying Equation 12.7. First notice that

$$?_{\xi\xi} = \frac{\mathrm{D}^2?}{\mathrm{D}\theta^2} - ??_x = \frac{\mathrm{D}\Psi}{\mathrm{D}\theta} - ??_x$$

as soon as

$$\Psi = \frac{\mathrm{D}?}{\mathrm{D}\theta} = ?_\theta + v?_x.$$

We compute the derivative of $E(t)$,

$$\begin{aligned} E'(t) &= \iint ??_{\xi\xi}\, dx d\theta \\ &= \iint ?(\Psi_\theta + v\Psi_x - ??_x)\, dx d\theta \\ &= \iint ?\Psi_\theta + (v?)\Psi_x - ?^2?_x\, dx d\theta. \end{aligned}$$

Integrating by parts the first two terms yields

$$E'(t) = \int [?\psi]_{\theta_1}^{\theta_2}\, dx + \int [v?\psi]_{-\infty}^{+\infty}\, d\theta - \iint ?_\theta\Psi + (v?)_x\Psi + ?^2?_x\, dx d\theta.$$

The first bracket is zero thanks to Equation 12.5 (or thanks to the periodicity of $?\Psi$ if $I = S^1$), and the second one is zero because $v?\Psi$ is compactly supported at any scale $t$. Hence, we have

$$\begin{aligned} E'(t) &= -\iint ?_\theta\Psi + (v?)_x\Psi + ?^2?_x\, dx d\theta \\ &= -\iint \Psi(?_\theta + v?_x + v_x?) + ?^2?_x\, dx d\theta \\ &= -\iint \Psi^2 dx d\theta - \iint v_x?\Psi\, dx d\theta + \iint ?^2?_x\, dx d\theta. \end{aligned} \tag{12.9}$$

But as

$$\iint ?^2?_x\, dx d\theta = \frac{1}{3} \iint \frac{\partial}{\partial x}(?^3)\, dx d\theta = 0$$

(because ? is compactly supported at any scale $t$), the second term of Equation 12.9 can be rewritten

$$\begin{aligned} B(t) = \iint v_x ? \Psi \, dx d\theta &= \iint v_x ? \Psi - ?^2 ?_x \, dx d\theta \\ &= \iint ? (?_\theta v_x + v v_x ?_x - ?_x v_\theta - v v_x ?_x) \, dx d\theta \\ &= \frac{1}{2} \iint (2? ?_\theta) v_x - (2? ?_x) v_\theta \, dx d\theta. \end{aligned}$$

then, another integration by parts yields

$$2B(t) = \int \left[ ?^2 v_x \right]_{\theta_1}^{\theta_2} dx + \int \left[ ?^2 v_\theta \right]_{-\infty}^{+\infty} d\theta - \iint ?^2 (v_{x\theta} - v_{\theta x}) \, dx d\theta = 0.$$

Finally, coming back to Equation 12.9, we obtain

$$E'(t) = - \iint \Psi^2 dx d\theta \leqslant 0$$

as announced. □

**Remark :** Since $E(t)$ is positive and decreases with scale, it converges to a minimum value as $t \to +\infty$, and $E'(t) \to 0$ as $t \to +\infty$. Now, what are the movies $u$ for which $\Psi = 0$ ? Coming back to the construction of the solutions of the DCMA, one easily checks that the condition $\Psi = 0$ is equivalent to the condition

$$\forall (x, \theta) \in \mathbb{R} \times I, \quad \varphi_{\theta\theta\theta}(x, \theta) = 0, \tag{12.10}$$

the map $\varphi$ being defined as usual by

$$u(\varphi(x, \theta), \theta) = u(x, 0).$$

Equation 12.10 implies the existence of three maps $A, B, C$ such that

$$\forall (x, \theta) \in \mathbb{R} \times I, \quad \varphi(x, \theta) = A(x)\theta^2 + B(x)\theta + C(x).$$

and since $\varphi(x, 0) = x$, necessarily $C(x) = x$. Hence, the level lines of a movie $u$ satisfying $\Psi = 0$ are parabolae.

### 12.5.2 A variational interpretation

At this point, it is natural to wonder whether Equation 12.8 results from a variational principle. Let us consider the functional

$$\mathcal{E}(v) = \frac{1}{2} \iint (v_\theta + v v_x)^2 \, dx d\theta,$$

defined on compactly supported movies of class $C^2$. Then, we can differentiate $\mathcal{E}$ to obtain

$$\begin{aligned} D_v\mathcal{E}(h) &= \iint (v_\theta + vv_x)(h_\theta + (vh)_x)\, dxd\theta \\ &= \iint ?\, h_\theta + (?\, v)h_x + ?\, v_x h\, dxd\theta. \end{aligned}$$

By integrating by parts the first two terms, we get, assuming that Equation 12.5 is satisfied by ? if $I \neq S^1$,

$$D_v\mathcal{E}(h) = \iint -?_\theta h - ?_x vh\, dxd\theta = -\iint \frac{\mathrm{D}?}{\mathrm{D}\theta} h\, dxd\theta,$$

that is to say

$$D_v\mathcal{E}(h) = -\iint \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} \cdot h\, dxd\theta.$$

Hence, the canonical evolution equation associated to the variational problem of minimizing $\mathcal{E}$ would be

$$\frac{\partial v}{\partial t} = \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} = v_{\xi\xi} + ?\, v_x.$$

Because of the last term $?\, v_x$, we can see that the equation $v_t = v_{\xi\xi}$ induced by the DCMA is not exactly the evolution equation associated to the minimization of $\mathcal{E}$. However, Proposition 41 showed that for the DCMA evolution,

$$D_v\mathcal{E}(\frac{\partial v}{\partial t}) = \frac{d}{dt}E(t) = -\iint \left(\frac{\mathrm{D}^2 v}{\mathrm{D}\theta}\right)^2 dxd\theta$$

as if it was the case[1]. Hence, the DCMA is somewhat related to the problem of minimizing $\mathcal{E}$.

## 12.6 Interpretation for the observed scene

In this section, we do not omit the $y$ variable any longer.

### 12.6.1 Ideal movies

**Definition 31** *A movie $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$ is **ideal** if one can find three maps $(C, Z, U) \in C^0(I^\star) \times C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2)$ such that*

$$\begin{aligned} \Pi \quad : \quad \mathbb{R}^2 \times \overline{I^\star} &\to \mathbb{R}^2 \times \overline{I} \\ (X, Y, \theta) &\mapsto \left(\frac{X - C(\theta)}{Z(X,Y)}, \frac{Y}{Z(X,Y)}, \theta\right) \end{aligned}$$

*is bijective and*

$$\forall (X, Y, \theta) \in \mathbb{R}^2 \times \overline{I^\star}, \qquad u \circ \Pi(X, Y, \theta) = U(X, Y). \tag{12.11}$$

[1]The reason is simply that

$$\iint \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} \Gamma v_x\, dxd\theta = 0$$

as we noticed previously.

In other terms, a movie is ideal if it can be interpreted as the perfect observation of a scene $Z(X,Y), U(X,Y)$ (depth and Lambertian luminance) by a unit focal length camera submitted to the movement $X = C(\theta)$. In this definition, occlusions are forbidden because $\Pi$ is constrained to be bijective. If $I = S^1$, the natural injection $\mathbb{R} \hookrightarrow S^1$ is implicit in the definition.

It is important to notice that the interpretation of a movie is never unique. Indeed, if $(C, Z, U)$ is an interpretation of $u$, then $(\lambda C, \lambda Z \circ D_\lambda, U \circ D_\lambda)$ with $D_\lambda : (X,Y) \mapsto (X/\lambda, Y/\lambda)$ is another interpretation of $u$. This ambiguity is called the aperture problem : if one do not know the focal length of a camera, the depth on the movie it produces can at most be recovered up to a multiplicative factor . Moreover, it is clear that the depth cannot be recovered in regions $(X, Y, Z(X,Y))$ where $U$ is constant. Ambiguities in the depth recovery can also appear in case of special relations between the depth (or luminance) and the camera movement, which are actually not likely to occur in practice (see [44]).

### 12.6.2 Differential characterization of ideal movies

**Proposition 42** *If a movie is ideal and allows a derivable movement interpretation, then it admits a velocity map $v$, and in any point where $v$ is $C^2$ we have*

$$v \cdot \nabla ? - ? \cdot \nabla v = \mathit{0}. \tag{12.12}$$

In Equation 12.12 the symbol $\nabla$ means the spatial gradient operator

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}),$$

and as usual

$$? = \frac{\mathrm{D}v}{\mathrm{D}\theta} = v_\theta + v v_x.$$

Hence, Equation 12.12 can also be rewritten

$$\begin{cases} v v_{\theta x} + v^2 v_{xx} - v_\theta v_x = 0 \\ v v_{\theta y} + v^2 v_{xy} - v_\theta v_y = 0 \end{cases}$$

***Proof :***

Let $(C, Z, U)$ be an interpretation of $u$ such that $C$ is of class $C^1$. We define a unique movie $v : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$ by

$$v \circ \Pi(X, Y, \theta) = \frac{-C'(\theta)}{Z(X,Y)}. \tag{12.13}$$

Then, differentiating Equation 12.11 with respect to $\theta$ yields

$$(v u_x + u_\theta) \circ \Pi = 0,$$

so that $v$ is a velocity map of $u$ as announced. Now, anywhere $v$ is $C^2$ we have

$$(v v_x + v_\theta) \circ \Pi(X, Y, \theta) = \frac{-C''(\theta)}{Z(X,Y)},$$

which can be combined with Equation 12.13 to yield

$$C'(\theta)\cdot ?\circ\Pi(X,Y,\theta)=C''(\theta)\cdot v\circ\Pi(X,Y,\theta)$$

because $Z$ does not vanish. Now, if $C'(\theta)\neq 0$, then $v\neq 0$ and $?/v$ does not depend on $x$, so that

$$0=\nabla\frac{?}{v}=\frac{v\nabla ?-?\nabla v}{v^2}$$

as announced. If $C'(\theta)=0$ and $C''(\theta)\neq 0$, the same reasoning applies to the map $v/?$. Last, if $C'(\theta)=0$ and $C''(\theta)=0$, Equation 12.12 is clearly satisfied because $v=?=0$. □

A natural question arises : does an ideal movie remain ideal when it evolves according to the DCMA ? To prove that the answer is yes, we could show that the differential invariant of Equation 12.12 remains null if it is null at initial scale. In fact, we state a better property by interpreting the evolution of an ideal movie.

### 12.6.3 Evolution of ideal movies

**Theorem 10** *Let $u_0\in\mathcal{C}^2_c$ be an ideal movie associated with an interpretation $(Z_0(\cdot),U_0(\cdot),C_0(\cdot))$ such that*

$$\exists A,B,\ \forall\theta\in I^\star,\quad |C_0(\theta)|\leqslant A+B|\theta|.$$

*Then the classical solution $u$ of the DCMA defined from the initial datum $u_0$ is a multi-scale collection of ideal movies $((u(\cdot,t))_{t\geqslant 0}$. Moreover, these movies can be interpreted as $(Z_0(\cdot),U_0(\cdot),C(\cdot,t))$, where $C(\cdot,\cdot)$ is defined by*

$$C_t=C_{\theta\theta}\quad\text{on}\quad \Omega=I^\star\times]0,+\infty[$$

*with the boundary condition*

$$\forall(\theta,t)\in\partial\Omega,\qquad C(\theta,t)=C_0(\theta).$$

***Proof :***

1. The movie $u_0$ being ideal, we have

$$\forall(X,Y,\theta)\in\mathbb{R}^2\times\overline{I},\quad u_0\left(\frac{X-C_0(\theta)}{Z_0(X,Y)},\frac{Y}{Z_0(X,Y)},\theta\right)=U_0(X,Y).$$

Let $C$ be the solution of the heat equation as specified in the theorem. The map

$$\begin{array}{rccc}\Pi & : & \mathbb{R}^2\times\overline{I^\star} & \to & \mathbb{R}^2\times\overline{I}\times[0,+\infty[\\ & & (X,Y,\theta,t) & \mapsto & \left(\dfrac{X-C(\theta,t)}{Z_0(X,Y)},\dfrac{Y}{Z_0(X,Y)},\theta,t\right)\end{array}$$

is bijective because the heat equation satisfies the comparison principle. Hence, we can define a collection of ideal movies $\tilde{u}(\cdot, t)$ from

$$\tilde{u} \circ \Pi(X, Y, \theta) = U_0(X, Y), \tag{12.14}$$

2. First we check that $\tilde{u}$ is $C^2$. Choose $(X_0, Y_0, \theta_0, t_0) \in \mathbb{R}^2 \times \overline{I^\star} \times ]0, +\infty[$, and write $(X(h), Y(h))$ the unique element of $\mathbb{R}^2$ such that

$$\Pi(X(h), Y(h), \theta_0, t_0) = \Pi(X_0, Y_0, \theta_0, t_0) + (h, 0, 0) = (x_0 + h, y_0, \theta_0, t_0).$$

We have, for any $\theta$ and $h$,

$$\tilde{u}(x_0 + h, y_0, \theta_0, t_0) = U_0(X(h), Y(h)) = u_0(x_0 + h + \frac{C(\theta_0, t_0) - C_0(\theta)}{Z_0(X(h), Y(h))}, y_0, \theta).$$

Now, there exists a unique $\theta_1$ such that

$$C_0(\theta_1) = C(\theta_0, t_0),$$

so that we finally get

$$\tilde{u}(x_0 + h, y_0, \theta_0, t_0) = u_0(x_0 + h, y_0, \theta_1).$$

This proves that $\tilde{u}$ is, like $u_0$, derivable with respect to $x$. A similar reasoning establishes that $u \in \mathcal{C}_c^{2,1}$.

3. Now we prove that $u = \tilde{u}$. If we compute the derivatives of Equation 12.14 with respect to $\theta$ and $t$, we obtain

$$-\frac{C'(\theta, t)}{Z_0(X, Y)} \tilde{u}_x \circ \Pi + \tilde{u}_\theta \circ \Pi = 0 \quad \text{and} \quad -\frac{C''(\theta, t)}{Z_0(X, Y)} \tilde{u}_x \circ \Pi + \tilde{u}_t \circ \Pi = 0.$$

If $\tilde{u}_x \circ \Pi = 0$, then $\tilde{u}_t \circ \Pi = 0$, and if $\tilde{u}_x \circ \Pi \neq 0$, eliminating $C$ yields

$$\tilde{u}_t \circ \Pi = \frac{\tilde{u}_x \circ \Pi}{Z_0(X, Y)} \frac{\partial}{\partial \theta} \left( Z_0(X, Y) \frac{\tilde{u}_\theta}{\tilde{u}_x} \circ \Pi \right) = \left[ \tilde{u}_x \frac{\mathrm{D}}{\mathrm{D}\theta} (\frac{\tilde{u}_\theta}{\tilde{u}_x}) \right] \circ \Pi = \tilde{u}_{\xi\xi} \circ \Pi.$$

Hence, $\tilde{u}$ is a classical solution of the DCMA submitted to the same boundary constraint as $u$. Since these conditions define a unique solution, we can deduce that $u = \tilde{u}$, which proves that each movie $u(\cdot, \cdot, \cdot, t)$ is ideal and that we can choose the interpretation announced in the theorem. □

The signification of Theorem 10 is simple : when analyzed by the DCMA, an ideal movie remains ideal and its interpretation is preserved up to a smoothing process on the camera movement.

### 12.6.4 Characterization of the DCMA

We now give another justification for the DCMA equation obtained in Theorem 9.

**Theorem 11** *The DCMA is, up to a rescaling, the only*[2] *multiscale analysis satisfying the architectural axioms, the* ***[v-compatibility]*** *axiom, and such that an ideal movie* $(C_0, Z, U)$ *is transformed into a sequence of ideal movies* $(C(t), Z, U)$ *such that* $C(t)$ *depends linearly on* $C_0$.

***Proof :***

1. Let us start by writing the relations between the scene referential $(X, Y, \Theta, T)$ and the image referential $(x, y, \theta, t)$ :

$$\begin{aligned} X &\leftrightarrow x = \frac{X - C(\theta,t)}{Z(X,Y,T)} \\ Y &\leftrightarrow y = \frac{Y}{Z(X,Y,T)} \\ \Theta &\leftrightarrow \theta \\ T &\leftrightarrow t \end{aligned}$$

From this, we compute the differentials

$$\begin{aligned} dx &= \frac{1 - xZ_X}{Z}dX - \frac{xZ_Y}{Z}dY - \frac{V}{Z}d\Theta - \frac{1}{Z}(C_t + xZ_T)dT \\ dy &= -\frac{yZ_X}{Z}dX - \frac{1 - yZ_Y}{Z}dY - \frac{y}{Z}Z_T dT \end{aligned}$$

Now, given a map $F$ defined on both referentials, we have

$$dF = F_X dX + F_Y dY + F_\Theta d\Theta + F_T dT = F_x dx + F_y dy + F_\theta d\theta + F_t dt,$$

so that

$$F_X = F_x\left(\frac{1 - xZ_X}{Z}\right) + F_y\left(\frac{-yZ_X}{Z}\right) \tag{12.15}$$

$$F_Y = F_x\left(\frac{-xZ_Y}{Z}\right) + F_y\left(\frac{1 - yZ_Y}{Z}\right) \tag{12.16}$$

$$F_\Theta = F_x\left(\frac{-V}{Z}\right) + F_\theta \tag{12.17}$$

$$F_T = F_x\left(-\frac{C_t}{Z}\right) - \frac{Z_T}{Z}(xF_x + yF_y) + F_t. \tag{12.18}$$

Notice that Equation 12.17 simply gives the total derivative of $F$,

$$\frac{\mathrm{D}F}{\mathrm{D}\theta} = F_\Theta = F_\theta + vF_x.$$

Now, applying Equation 12.15 and 12.16 to $F = Z$, we get

$$\begin{cases} ZZ_X = Z_x(1 - xZ_X) + Z_y(-yZ_X) \\ ZZ_Y = Z_x(-xZ_Y) + Z_y(1 - yZ_Y) \end{cases}$$

[2] Once again, the identity operator is naturally irrelevant here.

which yields, when the denominators do not vanish,

$$Z_X = \frac{Z_x}{Z + xZ_x + yZ_y} \quad \text{and} \quad Z_Y = \frac{Z_y}{Z + xZ_x + yZ_y}.$$

Using Equation 12.18 applied to $Z$, we finally obtain

$$Z_T = \frac{-Z_xC_t + ZZ_t}{Z + xZ_x + yZ_y} = Z_X\left(\frac{ZZ_t}{Z_x} - C_t\right).$$

2. Consider a multiscale analysis satisfying the hypotheses of Theorem 11. Then, from Lemma 17 we know that it can be described by an evolution equation of the kind

$$u_t = u_\theta F(\frac{?}{v}, v), \tag{12.19}$$

provided that we suppose that $v$ does not vanish. If $u$ is an ideal movie, we have $v = -V/Z$ if we note $V = C_\theta$, and Equation 12.19 can be rewritten

$$\frac{\partial u}{\partial t} = u_\theta F(\frac{V_\theta}{V}, -\frac{V}{Z}).$$

Then we can compute

$$v_t = -\frac{1}{u_x}\frac{\mathrm{D}u_t}{\mathrm{D}\theta} = v_\theta F + v\frac{VV_{\theta\theta} - V_\theta^2}{V^2}F_1 - v\frac{V_\theta}{Z}F_2, \tag{12.20}$$

$F_1$ and $F_2$ meaning the partial derivatives of $F$ with respect to arguments 1 and 2. Now, as

$$\frac{v_\theta}{v} = \frac{V_\theta}{V} - \frac{Z_\theta}{Z} = \frac{V_\theta}{V} - \frac{VZ_x}{Z^2}$$

and

$$\frac{v_t}{v} = \frac{V_t}{V} - \frac{Z_T}{Z},$$

Equation 12.20 yields

$$\frac{V_t}{V} - \frac{Z_T}{Z} = (\frac{V_\theta}{V} - \frac{VZ_x}{Z^2})F + \frac{VV_{\theta\theta} - V_\theta^2}{V^2}F_1 - \frac{V_\theta}{Z}F_2$$

Since the multiscale analysis must preserve the depth interpretation of the scene, we must have $Z_T = 0$, that is to say

$$\frac{Z}{Z_x}Z_t = C_t \quad \text{whenever} \quad Z_X \neq 0,$$

from what we deduce

$$\frac{V_t}{V} - \frac{V_\theta}{V}F - \frac{VV_{\theta\theta} - V_\theta^2}{V^2}F_1 - \frac{V_\theta}{Z}F_2 = \frac{Z_x}{Z^2}(C_t - VF). \tag{12.21}$$

The left term of Equation 12.21 only depends on $Z$, $\theta$, and $t$. Therefore, by formal independency of $Z_x$ we necessarily have $C_t = VF$ and $F$ only depends on $\theta$ and $t$, that is to say $F_2 = 0$. Then, Equation 12.21 is equivalent to

$$V_t = (V_{\theta\theta} - \frac{V_\theta^2}{V})F_1 + V_\theta F,$$

and the only possibility for $V$ to evolve linearly is

$$V_t = \lambda V_\theta,$$

which yields a trivial evolution equation on $u$, and $V_t = \lambda V_{\theta\theta}$, that is to say

$$F(a, b) = \lambda a.$$

This corresponds to the announced evolution Equation $u_t = u_{\xi\xi}$, up to the rescaling $t \mapsto \lambda t$. $\square$

# Chapter 13

# Numerical scheme and experiments

In this chapter, we propose a simple morphological scheme to implement the DCMA numerically. We prove its consistency in the "regular case", and investigate its behaviour when singularities appear. We link these observations with the difficulty encountered when trying to obtain theoretical existence properties for general initial data. Last, we present experiments on two classical movies of outdoor scenes, and we highlight both time regularization effects of the DCMA and its usefulness for depth recovery.

## 13.1 Definition

In order to apply the DCMA evolution to real movies, we need to devise a numerical scheme. A "naive" discretization of the partial derivatives of $u$ cannot be used, because in practice it is well known that the time discretization is not thin enough. Moreover, such a discretization is not likely to satisfy the axioms that we imposed to the DCMA. This is the reason why we focus our attention on an inf-sup scheme. To this end, given a movie $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$, we define

$$\begin{aligned} IS_h u(x_0, y_0, \theta_0) &= \inf_{v \in \mathbb{R}} \sup_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ SI_h u(x_0, y_0, \theta_0) &= \sup_{v \in \mathbb{R}} \inf_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ \text{and} \quad T_h u &= \frac{1}{2} \left( IS_h u + SI_h u \right). \end{aligned}$$

If $I = S^1$, all the quantities above are well defined. If $I = ]\theta_1, \theta_2[$, we take the convention that

$$u(x, y, \theta) = \begin{cases} u(x, y, \theta_1) & \text{for} \quad x < \theta_1, \\ u(x, y, \theta_2) & \text{for} \quad x > \theta_2. \end{cases}$$

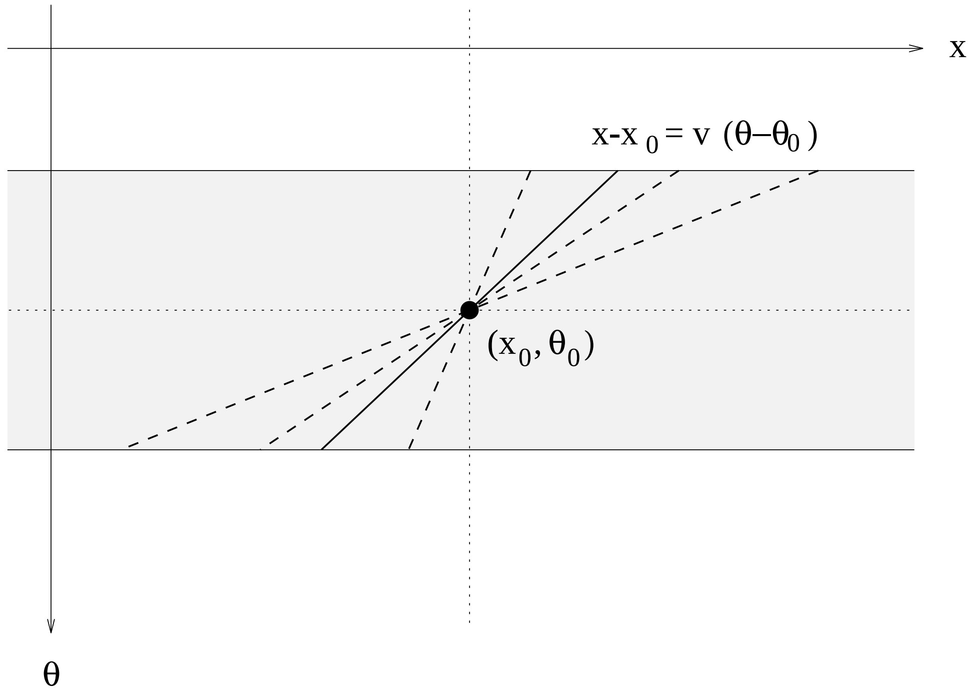


Figure 13.1: Inf-sup scheme used to implement the DCMA.

## 13.2 Consistency (regular case)

First, we establish a consistency result at points where $u_x$ does not vanish.

**Theorem 12** *If $u$ is a bounded movie locally $C^3$ near $\boldsymbol{z}_0$, with $u_x(\boldsymbol{z}_0) \neq 0$, then*

$$\begin{aligned} IS_h u(\boldsymbol{z}_0) &= u(\boldsymbol{z}_0) + h^2 u^+_{\xi\xi}(\boldsymbol{z}_0) + O(h^3), \\ SI_h u(\boldsymbol{z}_0) &= u(\boldsymbol{z}_0) + h^2 u^-_{\xi\xi}(\boldsymbol{z}_0) + O(h^3), \\ T_h u(\boldsymbol{z}_0) &= u(\boldsymbol{z}_0) + \frac{1}{2} h^2 u_{\xi\xi}(\boldsymbol{z}_0) + O(h^3). \end{aligned}$$

*and the $O(h^3)$ is uniform in a neighborhood of $\boldsymbol{z}_0$.*

From now on, we shall omit the $y$ variable in the movies we consider. Since most of the quantities involved in the following are continuous with respect to the $y$ variable, the corresponding estimations are easily proved to be locally uniform in the $y$ coordinate.

**Lemma 26** *Consider a bounded movie $u$ locally $C^2$ near $\boldsymbol{z}_0$ and such that $u_x(\boldsymbol{z}_0) \neq 0$. Then, in a neighborhood of $\boldsymbol{z}_0$ we have, for $h$ small enough,*

$$IS_h u = \tilde{IS}_h u,$$

$$\text{with} \qquad \tilde{IS}_h u : (x_1, \theta_1) \mapsto \inf_{|v| \leqslant \frac{1}{\sqrt{h}}} \sup_{|\theta| \leqslant h} u(x_1 + v\theta, \theta_1 + \theta).$$

***Proof :***

1. Let $K$ be a compact neighborhood of $\boldsymbol{z}_0$ on which $u$ is $C^2$ and $u_x$ does not vanish. We consider the compact set $K' = K + [-1,1] \times [-1,1]$, and write

$$A = \inf_K |u_x| \quad \text{and} \quad B = \sup_{K'} |u_x| + |u_\theta|.$$

From Taylor's Theorem, the map

$$C(x) = \sup_{(x_1,\theta_1)\in K} \frac{|u(x_1+x,\theta_1) - u(x_1,\theta_1) - x u_x(x_1,\theta_1)|}{x^2}$$

is upper-bounded on $[-1,1]$ by

$$\frac{1}{2}\sup_{K'} |u_{xx}|,$$

while on $[-\infty,-1] \cup [1,+\infty]$ we have

$$C(x) \leqslant \sup_{|x|\geqslant 1} \frac{2\|u\|_\infty + B|x|}{x^2} < \infty.$$

Therefore, writing $C = \|C\|_\infty$ yields

$$\forall (x_1,\theta_1) \in K,\ \forall x \in \mathbb{R}, \quad u(x_1+x,\theta_1) \geqslant u(x_1,\theta_1) + x u_x(x_1,\theta_1) - Cx^2.$$

For $h \leqslant 1$, let us take

$$x_2(x_1,\theta_1,h) = \frac{\text{sgn}\,(u_x(x_1,\theta_1))}{1+\frac{2C}{A}}\sqrt{h},$$

with the classical convention that

$$\text{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

We obtain

$$\begin{aligned} u(x_1+x_2,\theta_1) &\geqslant u(x_1,\theta_1) + \frac{\sqrt{h}}{1+\frac{2C}{A}}\left(|u_x(x_1,\theta_1)| - \frac{C\sqrt{h}}{1+\frac{2C}{A}}\right) \\ &\geqslant u(x_1,\theta_1) + \frac{\sqrt{h}}{1+\frac{2C}{A}}(A - \frac{A}{2}) \\ &\geqslant u(x_1,\theta_1) + D\sqrt{h} \quad \text{with} \quad D = \frac{A^2}{2A+4C} > 0. \end{aligned}$$

Moreover, as $|x_2| \leqslant \sqrt{h}$, we have

$$|v| \geqslant \frac{1}{\sqrt{h}} \quad \Rightarrow \quad \theta_2(x_1,\theta_1,h,v) := \frac{x_2}{vh} \in [-1,1] \quad \text{and} \quad (x_1 + v\theta_2 h, \theta_1 + \theta_2 h) \in K'.$$

Hence,

$$u(x_1+v\theta_2 h, \theta_1+\theta_2 h) \geqslant u(x_1+v\theta_2 h,\theta_1) - B|\theta_2|h \geqslant u(x_1,\theta_1) + D\sqrt{h} - Bh,$$

and taking $h_1 = \inf\left(1, \left(\frac{D}{2B}\right)^2\right)$ yields for any $(x_1, \theta_1) \in K$,

$$h \leqslant h_1, |v| \geqslant \frac{1}{\sqrt{h}} \quad \Rightarrow \quad s_h(v)(x_1,\theta_1) := \sup_{|\theta|\leqslant 1} u(x_1 + v\theta h, \theta_1 + \theta h) \geqslant u(x_1,\theta_1) + \frac{D}{2}\sqrt{h}. \quad (13.1)$$

2. For $(x_1, \theta_1) \in K$, we write

$$v_c(x_1,\theta_1) = \frac{-u_\theta}{u_x}(x_1,\theta_1) \quad \text{and} \quad f_{x_1,\theta_1}(x) = u(v_c x, x).$$

A second-order expansion of $f_{x_1,\theta_1}$ yields

$$\forall h \leqslant h_1,\ \forall (x_1,\theta_1) \in K,\ \forall (v,\theta) \in \mathbb{R} \times [-1,1], \quad u(x_1 + v\theta h, \theta_1 + \theta h) \leqslant u(x_1,\theta_1) + \frac{\theta^2 h^2}{2} E,$$

where

$$E = \sup_{(x_1,\theta_1)\in K, |x|\leqslant h_1} f''_{x_1,\theta_1}(x).$$

Thus, for $h \leqslant h_0 := \inf\left(h_1, \left(\frac{D}{E}\right)^{\frac{2}{3}}\right)$, Equation 13.1 yields

$$IS_h u(x_1,\theta_1) \leqslant u(x_1,\theta_1) + \frac{D}{2}\sqrt{h} \leqslant \inf_{|v|\geqslant \frac{1}{\sqrt{h}}} s_h(x_1,\theta_1),$$

which proves that

$$\forall h \leqslant h_0,\ \forall (x_1,\theta_1) \in K, \quad IS_h u(x_1,\theta_1) = \inf_{|v|\leqslant \frac{1}{\sqrt{h}}} s_h(v)(x_1,\theta_1) = I\tilde{S}_h u(x_1,\theta_1)$$

as expected. □

In the following Lemma, we equip the space $\mathbb{R}[X,\Theta]$ of 2-variables real polynomials with the norm given by the maximum of the absolute values of the coefficients, that is

$$\|\sum_{i,j} a_{ij} X^i \Theta^j\| = \max_{i,j} |a_{ij}|.$$

**Lemma 27** *Let $P(x,\theta)$ be a polynomial whose degree is at most two. If $P_x(0,0) \neq 0$, then*

$$\forall h \leqslant \frac{\inf(1, P_x(0,0)^2)}{16\|P\|^2 + 1}, \quad I\tilde{S}_h P(0,0) = P(0,0) + h^2 P^+_{\xi\xi}(0,0).$$

***Proof :***

1. Since the degree of $P$ is at most two, the second-order expansion of $P$ is exact :

$$P(v\theta,\theta) = P(0,0) + \theta(av+b) + \theta^2 (v,1)^T [D^2 P](v,1) \quad \text{with} \quad (a,b) = (P_x, P_\theta)(0,0).$$

Let us consider the new coordinate system $(av+b,\theta)$ instead of $(v,\theta)$ (this is valid because $a\neq 0$). Writing

$$Q(\delta)=(\frac{\delta-b}{a},1)^T[D^2P](\frac{\delta-b}{a},1),$$

we get

$$\begin{aligned} I\tilde{S}_hP(0,0) &= \inf_{|v|\leqslant\frac{1}{\sqrt{h}}}\ \sup_{-h\leqslant\theta\leqslant h} P(v\theta,\theta) \\ &= P(0,0)+\inf_{|v|\leqslant\frac{1}{\sqrt{h}}}\ \sup_{-h\leqslant\theta\leqslant h} \theta(av+b)+\theta^2(v,1)^T[D^2P](v,1) \\ &= P(0,0)+\inf_{|\frac{\delta\perp b}{a}|\leqslant\frac{1}{\sqrt{h}}}\ \sup_{-h\leqslant\theta\leqslant h} \theta\delta+\theta^2Q(\delta) \\ &= P(0,0)+\inf_{|\frac{\delta\perp b}{a}|\leqslant\frac{1}{\sqrt{h}}}\ \sup_{0\leqslant\theta\leqslant 1} \theta|\delta|h+\theta^2h^2Q(\delta). \end{aligned}$$

Now, let us define

$$u_h(\delta,\theta)=\theta h|\delta|+\theta^2h^2Q(\delta),\quad s_h(\delta)=\sup_{0\leqslant\theta\leqslant 1}u_h(\delta,\theta)\quad\text{and}\quad A_h=\inf_{|\frac{\delta\perp b}{a}|\leqslant\frac{1}{\sqrt{h}}}s_h(\delta).$$

We want to show that $A_h=h^2Q(0)^+$ for $h$ small enough.

2. For $h\leqslant\frac{a^2}{b^2}$, we have $|\frac{0-b}{a}|\leqslant\frac{1}{\sqrt{h}}$, so that

$$A_h=\inf_{|\frac{\delta\perp b}{a}|\leqslant\frac{1}{\sqrt{h}}}s_h(\delta)\leqslant s_h(0)=\sup_{0\leqslant\theta\leqslant 1}\theta^2h^2Q(0)=h^2Q(0)^+.$$

Besides, as $s_h(\delta)=\sup_{0\leqslant\theta\leqslant 1}u_h(\delta,\theta)\geqslant u_h(\delta,0)=0$, we know that

$$A_h=\inf_{\delta\in\mathbb{R}}s_h(\delta)\geqslant 0.$$

In particular, this proves that if $Q(0)\leqslant 0$, then $A_h=0=h^2Q(0)^+$.

3. Let us study the case $Q(0)>0$. One easily checks that $Q(\delta)$ is a polynomial with degree at most two, and that

$$\sup_{|\frac{\delta\perp b}{a}|\leqslant\frac{1}{\sqrt{h}}}|Q'(\delta)|\leqslant 2\|P\|\left(\frac{1}{\sqrt{h}}+\frac{1}{|a|}\right).$$

As a consequence, for $h\leqslant a^2$ we have

$$\begin{aligned} \forall\delta,\quad |\frac{\delta-b}{a}|\leqslant\frac{1}{\sqrt{h}} &\Rightarrow Q(\delta)\geqslant Q(0)-4\frac{\|P\|}{\sqrt{h}}|\delta| \\ &\Rightarrow \forall\theta\in[0,1],\quad \theta h|\delta|+\theta^2h^2Q(\delta)\geqslant\theta h|\delta|+h^2\theta^2(Q(0)-4\frac{\|P\|}{\sqrt{h}}|\delta|) \\ &\Rightarrow \forall\theta\in[0,1],\quad u_h(\delta,\theta)\geqslant\theta h|\delta|(1-4\|P\|\sqrt{h}\theta)+h^2\theta^2Q(0) \\ &\Rightarrow \sup_{\theta\in[0,1]}u_h(\delta,\theta)\geqslant\Big(\theta h|\delta|(1-4\|P\|\sqrt{h}\theta)+h^2\theta^2Q(0)\Big)_{\theta=1} \\ &\Rightarrow s_h(\delta)\geqslant h|\delta|(1-4\|P\|\sqrt{h})+h^2Q(0), \end{aligned}$$

and taking the inf on $\delta$ yields

$$\forall h \leqslant h_1 := \frac{1}{16\|P\|^2}, \qquad A_h = \inf_{|\frac{\delta \perp b}{a}| \leqslant \frac{1}{\sqrt{h}}} s_h(\delta) \geqslant h^2 Q(0).$$

4. We showed in 2 and 3 that $A_h = h^2 Q(0)^+$ as soon as $h \leqslant h_0$, with

$$h_0 := \frac{\inf(1, a^2)}{16\|P\|^2 + 1} \leqslant \inf\left(a^2, \frac{a^2}{b^2}, \frac{1}{16\|P\|^2}\right).$$

This achieves the proof since

$$\begin{aligned} \forall h \leqslant h_0, \quad I\tilde{S}_h P(0,0) &= P(0,0) + h^2 Q(0)^+ \\ &= P(0,0) + h^2 \left[ (\frac{-b}{a}, 1)^T [D^2 P] (\frac{-b}{a}, 1) \right]^+ \\ &= P(0,0) + h^2 P_{\xi\xi}^+(0,0). \end{aligned}$$

□

***Proof of Theorem 12 :***

Let $K$ be a compact neighborhood of $\boldsymbol{z}_0 = (x_0, \theta_0)$ on which $u$ is $C^3$ and $u_x$ does not vanish. For $(x_1, \theta_1) \in K$, we write $P_{x_1,\theta_1}$ the second-order expansion of $u$ near $(x_1, \theta_1)$. The regularity of $u$ ensures the existence of a constant $C > 0$ such that

$$\forall (x_1, \theta_1) \in K,\ \forall (x, \theta) \in [-1, 1]^2, \quad |u(x_1 + x, \theta_1 + \theta) - P_{x_1,\theta_1}(x, \theta)| \leqslant C\sqrt{x^2 + \theta^2}^3.$$

This implies, for $h \in [0, 1]$,

$$\forall |\theta| \leqslant 1,\ \forall |v| \leqslant \frac{1}{\sqrt{h}}, \quad |u(x_1 + v\theta h, \theta_1 + \theta h) - P_{x_1,\theta_1}(v\theta h, \theta h)| \leqslant C\sqrt{2}h^3. \tag{13.2}$$

From now on, we fix $(x_1, \theta_1) \in K$ and write $P$ for $P_{x_1,\theta_1}$. If we apply the nondecreasing operator $I\tilde{S}_h$ to Equation 13.2, we get

$$\forall h \in [0,1], \quad I\tilde{S}_h P(x_1, \theta_1) - C\sqrt{2}h^3 \leqslant I\tilde{S}_h u(x_1, \theta_1) \leqslant I\tilde{S}_h P(x_1, \theta_1) + C\sqrt{2}h^3.$$

Notice that the regularity of $u$ implies that the map

$$(x_1, \theta_1) \mapsto P_{x_1,\theta_1}$$

is continuous, as well as the map

$$P \mapsto \frac{\inf(1, P_x(0,0)^2)}{16\|P\|^2 + 1}.$$

Hence, Lemma 27 ensures the existence of a constant $h_1 > 0$ *independent of* $(x_1, \theta_1)$ such that

$$\forall h \in [0,1], \quad I\tilde{S}_h P(0,0) = P(0,0) + h^2 P_{\xi\xi}^+(0,0) = u(x_1, \theta_1) + h^2 u_{\xi\xi}^+(x_1, \theta_1).$$

In addition, from Lemma 26 we know that there exists $h_2$, independent of $(x_1, \theta_1)$, such that

$$\forall h \leqslant h_2, \quad IS_h u(x_1, \theta_1) = I\tilde{S}_h u(x_1, \theta_1).$$

Therefore, for any $h \leqslant h_0 := \inf(h_1, h_2)$ we have

$$IS_h(u) = u + h^2 u_{\xi\xi}^+ + O(h^3)$$

uniformly on $K$. The symmetric estimation on $SI_h$ arises from

$$SI_h(u) = -IS_h(-u) = u - h^2(-u_{\xi\xi})^+ + O(h^3) = u + h^2 u_{\xi\xi}^- + O(h^3),$$

and summing up these two estimations establishes the desired consistency property

$$T_h(u) = u + h^2 u_{\xi\xi} + O(h^3)$$

uniformly on $K$. □

Theorem 12 proves the consistency of the numerical scheme given by the iteration of $T_h$ with respect to the DCMA evolution. Due to the $h^2$ coefficient in the expansion of $T_h$, it is natural to consider the numerical scheme which associates, to a given movie $u_0$ and a scale $t \geqslant 0$, the sequence of movies $(u_{n,t})_{n \geqslant 1}$ given by

$$u_n = T_{h_n}^n u_0, \quad \text{with} \quad h_n = \sqrt{2t/n},$$

and satisfying the boundary constraint

$$\forall (x, y, \theta) \in \partial(\mathbb{R}^2 \times I), \qquad u_n(x, y, \theta) = u_0(x, y, \theta).$$

For an operator $T$, the notation $T^n$ means $T \circ T \circ \ldots \circ T$ $n$ times.

Thanks to Theorem 12, we know that such a scheme is consistent. As for the convergence, we could hope to prove that $u_n$ converges towards the DCMA of $u_0$ when the partial derivative of $u_0$ with respect to $x$ never vanishes (but this would no be very useful). Unfortunately, we do not think that this numerical scheme (or any other) converges towards a solution of the DCMA in the general case. Indeed, as we explained in Chapter 11, we believe that such a solution does not exist in general. We try to make clearer that point by investigating what happens near singular points, i.e. points where $u_x = 0$. Although the non-existence of general solutions for the DCMA is a real theoretical problem, in practice the convergence of the numerical scheme is assured due to the discrete nature of computer data (of course, the question of the interpretation of the limit then becomes more tricky).

## 13.3 Singular points

We first establish a preliminary lemma.

**Lemma 28** *If* $(a,b) \in I\!R \times [0,+\infty[$, *then*

$$F(a,b) := \sup_{0\leqslant\theta\leqslant 1} \frac{a}{2}\theta^2 + b\theta = \begin{cases} \frac{a}{2}+b & \text{if} \quad a \geqslant -b, \\ -\frac{b^2}{2a} & \text{if} \quad a < -b. \end{cases}$$

***Proof :***

The map

$$\varphi(\theta) = \theta \mapsto \frac{a}{2}\theta^2 + b\theta$$

is $C^1$ on the compact set $K = [0,1]$, so that it attains its maximum value on $K$ either on $\partial K$ or in a critical point. That is,

$$\sup_{0\leqslant\theta\leqslant 1} \varphi(\theta) = \max(\varphi(0), \varphi(1), \varphi(-\frac{b}{a})) = \max(0, \frac{a}{2}+b, -\frac{b^2}{2a}),$$

with the convention $-b/a = -b^2/(2a) = -\infty$ if $a = 0$. □

**Proposition 43** *Let $P$ be a polynomial with degree at most two such that $P_x(x_0,\theta_0) = 0$. Then, in $(x_0,\theta_0)$ we have, as $h \to 0$,*

$$T_h P = P + \frac{h}{2}\,|P_\theta|\,\mathrm{sgn}(P_{xx}) + O(h^2)$$

***Proof :***

Without loss of generality, we can suppose that $(x_0,\theta_0) = (0,0)$. Since the degree of $P$ is at most two, we have

$$P(v\theta h, \theta h) = P(0,0) + b\cdot\theta h + \frac{h^2\theta^2}{2}Q(v),$$

where

$$b = P_\theta(0,0) \quad \text{and} \quad Q(v) = P_{\theta\theta}(0,0) + 2vP_{\theta x}(0,0) + v^2 P_{xx}(0,0).$$

Therefore,

$$\begin{aligned}(IS_hP - P)(0,0) &= \inf_v \sup_{|\theta|\leqslant 1}\left(b\cdot\theta h + \frac{h^2\theta^2}{2}Q(v)\right) \\ &= h\cdot\inf_v \sup_{0\leqslant\theta\leqslant 1}\left(|b|\cdot\theta + \frac{h\theta^2}{2}Q(v)\right) \\ &= h\inf_v F(hQ(v), |b|) \\ &= hF(h\cdot\inf Q, |b|)\end{aligned}$$

because the map $a \mapsto F(a,b)$ is increasing.

- if $|b| = 0$ or $\inf Q = -\infty$, then

$$(IS_hP - P)(0,0) = \frac{h^2}{2}\left[\inf Q\right]^+.$$

- if $|b| \neq 0$ and $\inf Q > -\infty$, then $h \cdot \inf Q \geqslant -|b|$ for $h$ small enough, so that

$$(IS_hP - P)(0,0) = h \cdot |b| + \frac{h^2}{2}\inf Q + o(h^2).$$

Now, one can see easily that

(i) if $P_{xx}(0,0) > 0$, then $\inf Q = \left(P_{\theta\theta} - \frac{P_{\theta x}^2}{P_{xx}}\right)(0,0)$.

(ii) if $P_{xx}(0,0) = P_{\theta x}(0,0) = 0$, then $\inf Q = P_{\theta\theta}(0,0)$.

(iii) if $P_{xx}(0,0) < 0$ or $(P_{xx}(0,0) = 0$ and $P_{\theta x}(0,0) \neq 0)$, then $\inf Q = -\infty$.

Hence, in $(0,0)$ we have

$$IS_hP = P + O(h^2) + \begin{cases} h|P_\theta| & \text{if} \quad P_{xx} > 0 \ \text{ or } \ P_{xx} = P_{\theta x} = 0, \\ 0 & \text{else.} \end{cases}$$

Recalling that $SI_hP = -(IS_h(-P))$, we obtain

$$SI_hP = P + O(h^2) + \begin{cases} -h|P_\theta| & \text{if} \quad P_{xx} < 0 \ \text{ or } \ P_{xx} = P_{\theta x} = 0, \\ 0 & \text{else.} \end{cases}$$

and finally

$$T_hP = P + O(h^2) + \frac{1}{2}\cdot\begin{cases} h|P_\theta| & \text{if} \quad P_{xx} > 0, \\ -h|P_\theta| & \text{if} \quad P_{xx} < 0, \\ 0 & \text{else} \end{cases}$$

as expected. □

The following table gives the values of $IS_h$, $SI_h$ and $T_h$ up to order 2 in $h$ according to conditions on $P_\theta$, $P_{xx}$ and $P_{\theta x}$. All these equalities hold for $h$ small enough, and we took the convention that

$$P_{\xi\xi} := \begin{cases} P_{\theta\theta} & \text{if} \quad P_{xx} = P_{\theta x} = 0, \\ P_{\theta\theta} - \dfrac{P_{\theta x}^2}{P_{xx}} & \text{if} \quad P_{xx} \neq 0. \end{cases}$$

| $P_\theta$ | $P_{xx}$ | $P_{\theta x}$ | $IS_hP - P$ | $SI_hP - P$ | $T_hP - P$ |
|---|---|---|---|---|---|
| $= 0$ | $= 0$ | $= 0$ | $\frac{h^2}{2}P_{\xi\xi}^+$ | $\frac{h^2}{2}P_{\xi\xi}^-$ | $\frac{h^2}{4}P_{\xi\xi}$ |
| $= 0$ | $> 0$ | | $\frac{h^2}{2}P_{\xi\xi}^+$ | $0$ | $\frac{h^2}{4}P_{\xi\xi}^+$ |
| $= 0$ | $< 0$ | | $0$ | $\frac{h^2}{2}P_{\xi\xi}^-$ | $\frac{h^2}{4}P_{\xi\xi}^-$ |
| $\neq 0$ | $> 0$ | | $h\lvert P_\theta\rvert + \frac{h^2}{2}P_{\xi\xi}$ | $0$ | $\frac{h}{2}\lvert P_\theta\rvert + \frac{h^2}{4}P_{\xi\xi}$ |
| $\neq 0$ | $< 0$ | | $0$ | $-h\lvert P_\theta\rvert + \frac{h^2}{2}P_{\xi\xi}$ | $-\frac{h}{2}\lvert P_\theta\rvert + \frac{h^2}{4}P_{\xi\xi}$ |
| $\neq 0$ | $= 0$ | $= 0$ | $h\lvert P_\theta\rvert + \frac{h^2}{2}P_{\xi\xi}$ | $-h\lvert P_\theta\rvert + \frac{h^2}{2}P_{\xi\xi}$ | $\frac{h^2}{4}P_{\xi\xi}$ |
| | $= 0$ | $\neq 0$ | $0$ | $0$ | $0$ |

□

Proposition 43 shows that if $u_x$ happens to vanish when $u_\theta$ does not, then we can expect the numerical scheme to blow up because of the non-zero coefficient of $h$. In fact, in case the limit of $u_n(x,\theta,t)$ exists as $n \to +\infty$, it is not likely that it will be continuous in $t = 0$. The best we can expect is that

$$u_0 \mapsto \lim_{t\to 0} \lim_{n\to+\infty} u_n(t)$$

defines a kind of projection from $\mathcal{C}_c^0$ to $\mathcal{V}_c^0$. According to Proposition 43, this projection might be obtained by the asymptotic state as $t \to +\infty$ of the solution of the PDE

$$u_t = \begin{cases} |u_\theta|\mathrm{sgn}(u_{xx}) & \text{if} \quad u_x = 0, \\ 0 & \text{else.} \end{cases}$$

Of course, all of this is purely intuitive. Evans also predicted a projection property (see [32]) by considering the DCMA Equation as the limit when $\varepsilon \to 0$ of the more regular equation

$$u_t = \frac{u_x^2 u_{\theta\theta} - 2u_x u_\theta u_{x\theta} + u_\theta^2 u_{xx}}{u_x^2 + \varepsilon^2 u_\theta^2}. \tag{13.3}$$

In particular, Equation 13.3 admits viscosity solutions as a slightly modified version of the mean curvature motion. The difference is that Evans proved that when $u$ is the characteristic function of an S-shaped curve, his construction leads to a different projection operator, based on a Maxwell area construction (see Figure 13.2).

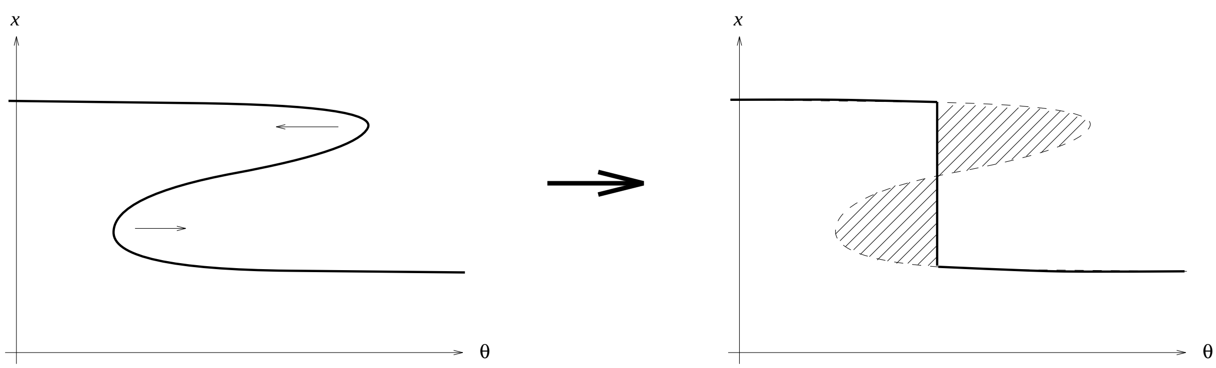


Figure 13.2: An S-shaped curve is immediatly transformed into a graph, the two dashed zones being of equal area (Maxwell equi-area construction). “The smoothing effects of the heat equation are so pronounced that a multi-valued data instantaneously unfolds into a graph” (Evans). The consequence for the DCMA is that solutions are not likely to exist for an initial datum whose level curves are not graphs. Indeed, such solutions could not be continuous at scale $t = 0$.

## 13.4 Algorithms

In this section, we describe the algorithms we used to experiment our analysis on numerical movies. These algorithms (and many others) will be available in the next public version of the MegaWave2 software, which can be freely[1] downloaded by anonymous ftp to the address ceremade.dauphine.fr or on the web server http://www.ceremade.dauphine.fr.

### 13.4.1 Data preparation

Even if a movie is realized in the conditions we described in introduction (that is to say, a straight translation of the camera parallel to its horizontal axis), in practice it is impossible to ensure that the camera movement has no vertical component at all. Hence, it is generally necessary to apply little vertical translations to the images of a real movie in order to compensate for the small vertical moves of the camera. Such an operation had already been performed (as explained in [13]) on the "TREES" movie we got from the SRI International Center. We needed to perform this operation on the "GARDEN" movie ourself (both these movies are presented later). The determination of these little vertical translations is not difficult since they affect all points of each image equally. In practice, it can be done by using a simple correlation measure. Such a simple algorithm is quite precise enough for our aim : in fact, we discovered later that an error of one pixel in a vertical movement compensation is immediatly overcome by the DCMA filtering.

### 13.4.2 Filtering with the DCMA

In order to experiment the effects of the DCMA, we need to discretize the numerical inf-sup scheme we described in the beginning of this chapter. The natural discrete choice for $h$ is $h = one\ image$, and in order to take into account the discrete nature of velocities it is also natural to consider discrete 3-points segments of the kind

$$\{(x-v,y,\theta-1),(x,y,\theta),(x+v+\varepsilon,y,\theta+1)\},$$

where all quantities are integer and $\varepsilon\in\{-1,0,1\}$. Hence, the discrete inf-sup operator is

$$ISu(x,y,\theta) = \min_{\substack{v\in\{-v_{max},...,-1,0\}\\ \varepsilon\in\{-1,0,1\}}} \max\{u(x-v,y,\theta-1),u(x,y,\theta),u(x+v+\varepsilon,y,\theta+1)\}.$$

The parameter $v_{max}$ must not be smaller than the largest velocity on the processed movie, which can easily be estimated. More important is the non-symmetric choice we made on $v$ by allowing only nonpositive velocities. There are several reasons for this choice : first, if the camera always goes forwards and never stops and goes back, then all velocities on the movie

[1]for non commercial use only, see the MegaWave2 documentation.

must be theoritically nonpositive. In addition, since the velocity field follows a causal evolution equation, it satisfies the maximum principle and is then forced to remain nonpositive at any scale of analysis. This proves the consistency of our non-symmetric choice of allowed velocities. Another reason that justifies this choice and that we shall discuss later is related to the filtering of occlusions.

The $SI$ and $IS$ operator being defined, we still have an alternative : either we iterate the mean operator $\frac{1}{2}(IS+SI)$ as we explained in the numerical scheme, or we iterate the alternated operator[2] $IS \circ SI$. No computational cost seems relevant to choose between the two possibilities, because it is roughly equivalent to compute $IS$ or simultaneously $IS$ and $SI$ on a movie, and one easily checks that one iteration of the alternated operator is also roughly equivalent, in terms of scale of analysis, to two iterations of the mean operator. In fact, when we tried both solutions, the advantage came to the alternated scheme, for two reasons.

The first reason is that it is purely morphological (and hence more consistent with our axiomatic formulation), with the consequence that no new grey-level is created when a movie is processed. This overcomes a purely numerical constraint : since the grey levels of a movie are practically discretized (typically, in $\{0,1,\ldots 255\}$ when represented by a 8-bit unsigned character), the division by two is not symmetrical and the result often has to be truncated, which has undesirable consequences after several iterations (notice that this cannot be avoided in practice by considering float values because of the huge amount of memory involved). Of course, the choice of an alternated operator is not symmetrical either (you can choose $IS \circ SI$ or $SI \circ IS$), but there are many less consequences.

The second reason is that a pure morphological scheme was more adapted to the algorithm we chose in order to compute the velocity field on the movie. This will become clear in the next section.

It is important to notice the extreme simplicity of the algorithm we presented : in particular, it can be implemented very easily on a massive parallel machine. Our optimized code in C language for one iteration consists of only 23 instructions.

### 13.4.3 Computing velocities

Of course, since the DCMA is devoted to the depth recovery — or, equivalently, to the computation of the velocity field —, it would not be enough to show filtered movies without checking the consequences of the DCMA on their inherent velocity fields. For that reason, we need to devise an algorithm to compute such velocity fields. Now comes the great interest of the DCMA : since the multiscale analysis theoretically produces a perfect time-coherent movie, we can use a naive algorithm to compute the velocity field.

[2] Though we did not prove explicitely the consistency of the alternated operator, it seems rather clear if we compare it to classical related schemes.

Our algorithm is global and takes only one parameter : the number $n$ of matching images we require to decide that a velocity is reliable. Given a point $(x_0, y_0, \theta_0)$, we look for the maximum value of $k$ for which there exist two real numbers $v_1$ and $v_2$ satisfying

$$-v_{max} \leqslant v_1 \leqslant v_2 \leqslant 0$$

and such that[3]

$$\forall \theta \in \{\theta_0, \dots, \theta_0 + k\}, \forall x \in \{E(x_0 - v_1\theta), \dots, E(x_0 - v_2\theta)\}, \qquad u(x, y_0, \theta) = u(x_0, y_0, \theta_0).$$

Then, we decide that the velocity field in $(x_0, y_0, \theta_0)$ is non-computable if $k < n$, and equal to $v_1$ if $k \geqslant n$ (of course, the interval $[v_1, v_2]$ is supposed to have a maximal length). The choice of $v_1$ (instead of $\frac{1}{2}(v_1 + v_2)$ for example) is logical but not very important since in practice we almost always have $v_1 \simeq v_2$ For symmetry, we also look for matchings in "past" times $\{\theta_0 - k, \dots, \theta_0\}$.

[3]here, the function $E()$ means the rounded integer part, that is to say $E(x) = n \in \mathbb{N} \Leftrightarrow n - \frac{1}{2} < x \leqslant n + \frac{1}{2}$.

## 13.5 Experiments

### 13.5.1 TREES movie (natural)

We picked up the “TREES” movie used by the SRI center (see [13]) by anonymous ftp to the adress periscope.cs.umass.edu. We obtained 64 images of size 256x233, which represent an amount of data of 3.8 Mo. According to [13], this movie is supposed to contain 128 images, but we could not find the remaining images ; however, 64 images were quite enough to test our algorithm.

Since the images were very dark, we first applied an optimal contrast change[4] to the movie : this process has only visual consequences thanks to the pure morphological invariance of our algorithms.

As we said before, this movie did not require a compensation for small vertical movements of the camera (it had been already done according to [13]).

Each iteration of the DCMA filter took 24 seconds. This represents a processing speed of about 0.16 Mo/s.

This movie is not the best choice to highlight the good properties of the DCMA, because of the strong occlusion caused by the foreground tree (we remind that our theory does not handle with occlusions). This occlusion caused smudging effects on the right side of this tree (and not on the left side thanks to the nonpositiveness of allowed velocities). However, these bad effects excepted, the algorithm proved to behave very well. The first striking visual effect of the algorithm on this movie is the strong time-coherence induced on the movie : it looks like all images become exactly equivalent except that the relative velocities of objects differ. In particular, there were important global intensity fluctuations between images on the initial movie : such a defect was completely removed by the DCMA. One could object that this regularization is paid by a visual loss of details on the ground texture. This is true and very logical since all non-time-coherent details cannot be preserved by the analysis. Although the DCMA has theoretically no spatial regularizing effects, such a spatial regularization actually occurs as a consequence of the time regularization.

[4]Applying a contrast change consists of modifying an initial movie $u$ into the movie $g \circ u$, where $g$ is an increasing grey-level correspondance map. It is said to be optimal if the histogram of the resulting movie is as flat as possible (which means that the grey levels are “used” in the best possible manner).

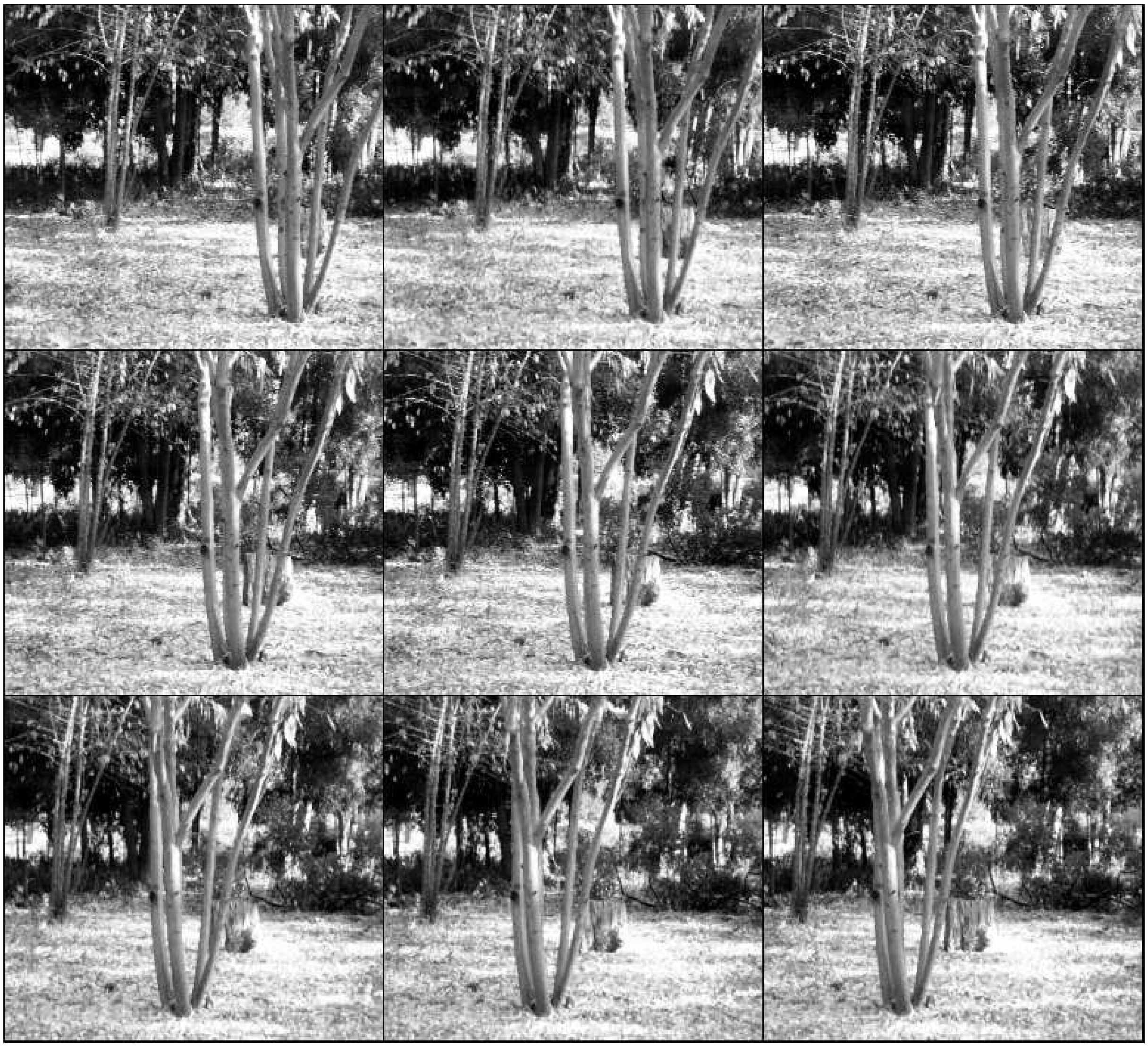

Figure 13.3: Original "TREES" movie.

*From left to right and then top to bottom : images number 1, 9, 17, 25, 32, 40, 48, 56 and 64 of the "TREES" movie (made of 64 image). The camera has a straight translation movement parallel to the horizontal axis of the image plane, and moving to the right. The relative positions of objects vary due to their different distances from the image plane (the closer they are, the quicker they "move" on the image).*

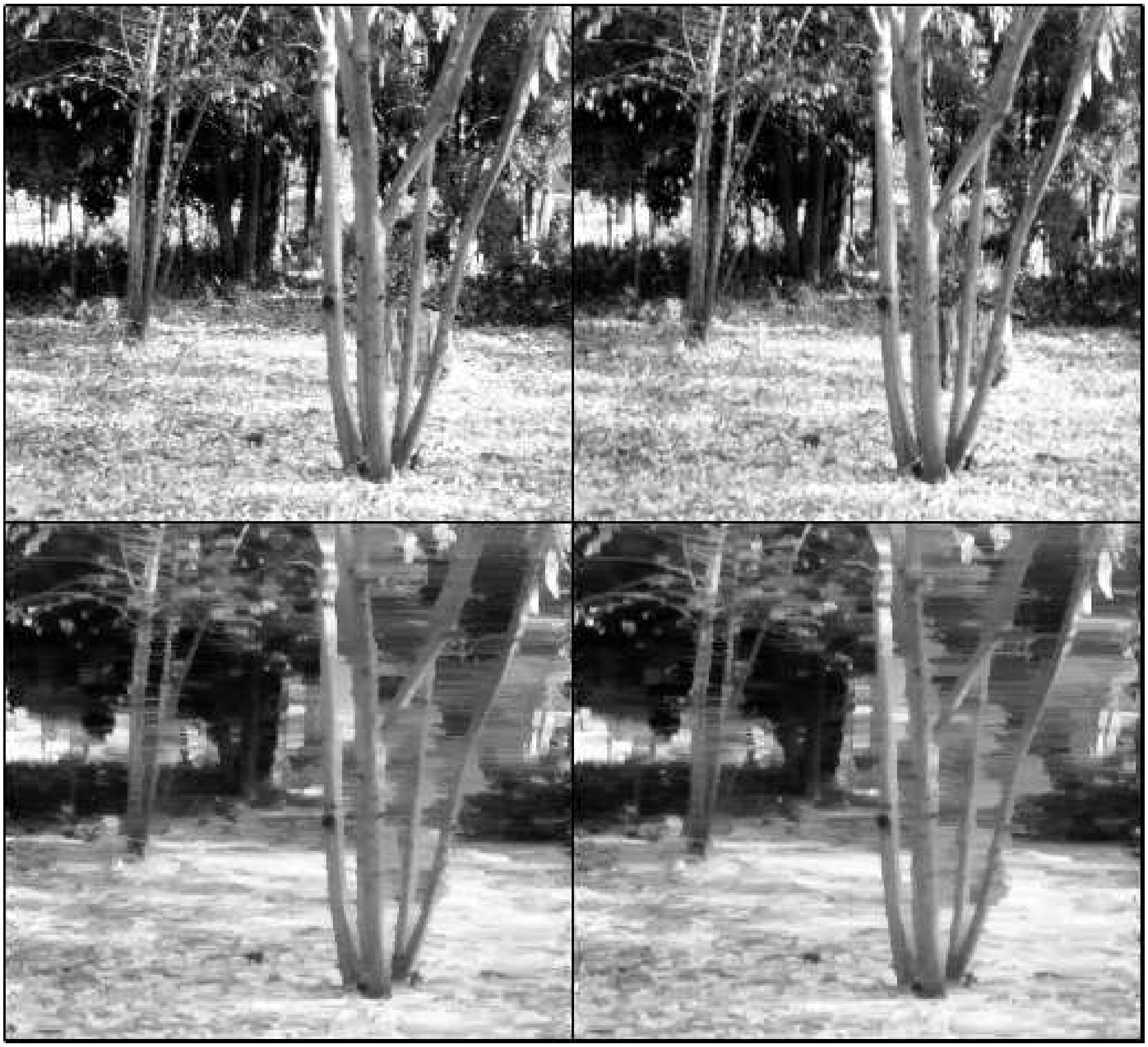

Figure 13.4: Filtering of the "TREES" movie.

*Top row :* *images 18 (left) and 22 (right) of the original "TREES" movie*
*Bottom row :* *images 18 and 22 of the "TREES" movie processed with 31 iterations.*

*The original movie has small details which cannot be tracked between successive images (they are not time-coherent), because the Nyquist limit for the time frequencies has been exceeded during the sampling process. The strong smoothing effects of the analysis (on the ground for example) are necessary to ensure the time coherence of the movie. The smudging effects near the branches of the foreground tree, however, are undesired and due to the incapacity of the DCMA to handle occlusions.*

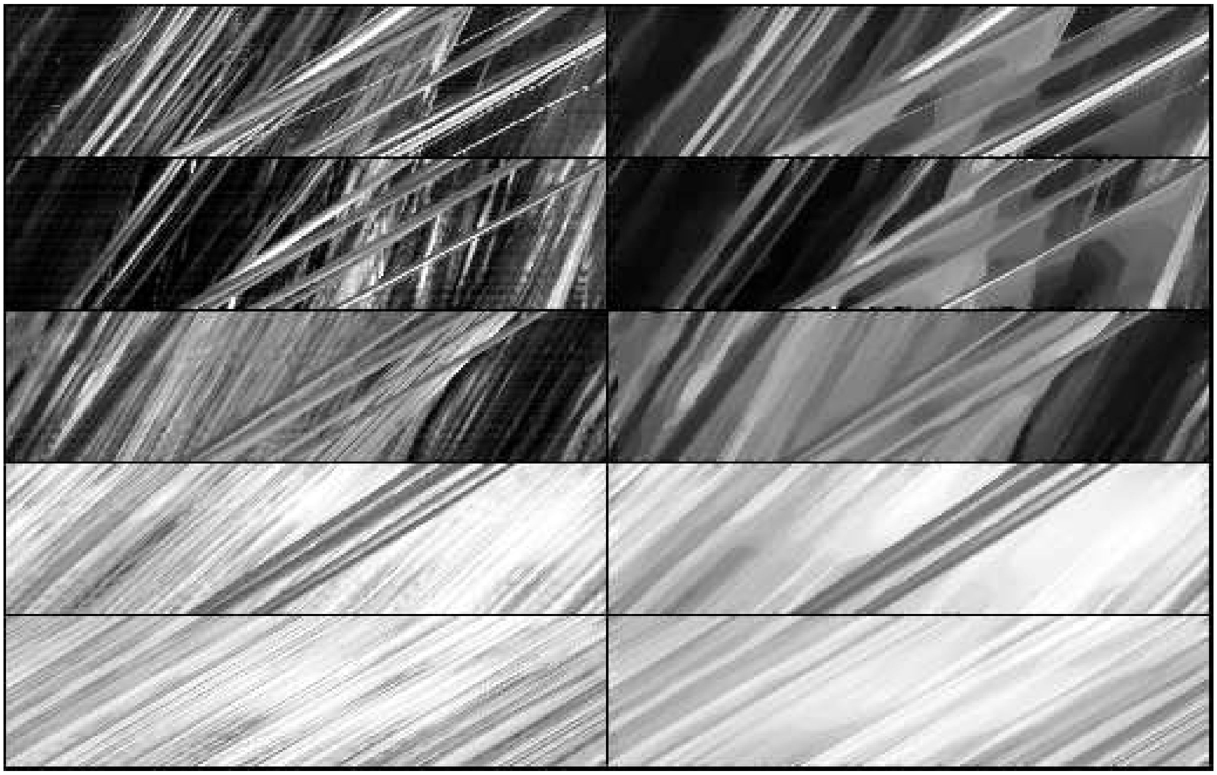

Figure 13.5: Analysis of the epipolar images.

*The epipolar images are obtained by slicing the movie $u(x,y,\theta)$ along $(x,\theta)$ planes for fixed values of $y$. The resulting images $(x,\theta) \mapsto u(x,y,\theta)$ are represented as follows : the $x$ axis is taken horizontal and the time axis $\theta$ is taken vertical pointing downwards. The epipolar images on colum 1 are taken from the original "TREES" movie (the values of $y$ are 20, 60, 140, 180, 220 respectively for rows 1, 2, 3, 4, 5). Those on column 2 are obtained after processing the original ones with 31 iterations.*

*Remember that the DCMA operates independently on all these epipolar images. The level lines of these images tend to become straight lines when analyzed by the DCMA ; a consequence is that the time-coherence of the analyzed movie increases with scale. On the original epipolar images, occlusions appear when two lines intersect : only the one with the smallest slope (i.e. representing the object closest to the camera) remains during the occlusion, the other one being occluded. Notice that occluded objects are often destroyed by the DCMA (see row 2 for example), because the DCMA cannot handle occlusions.*

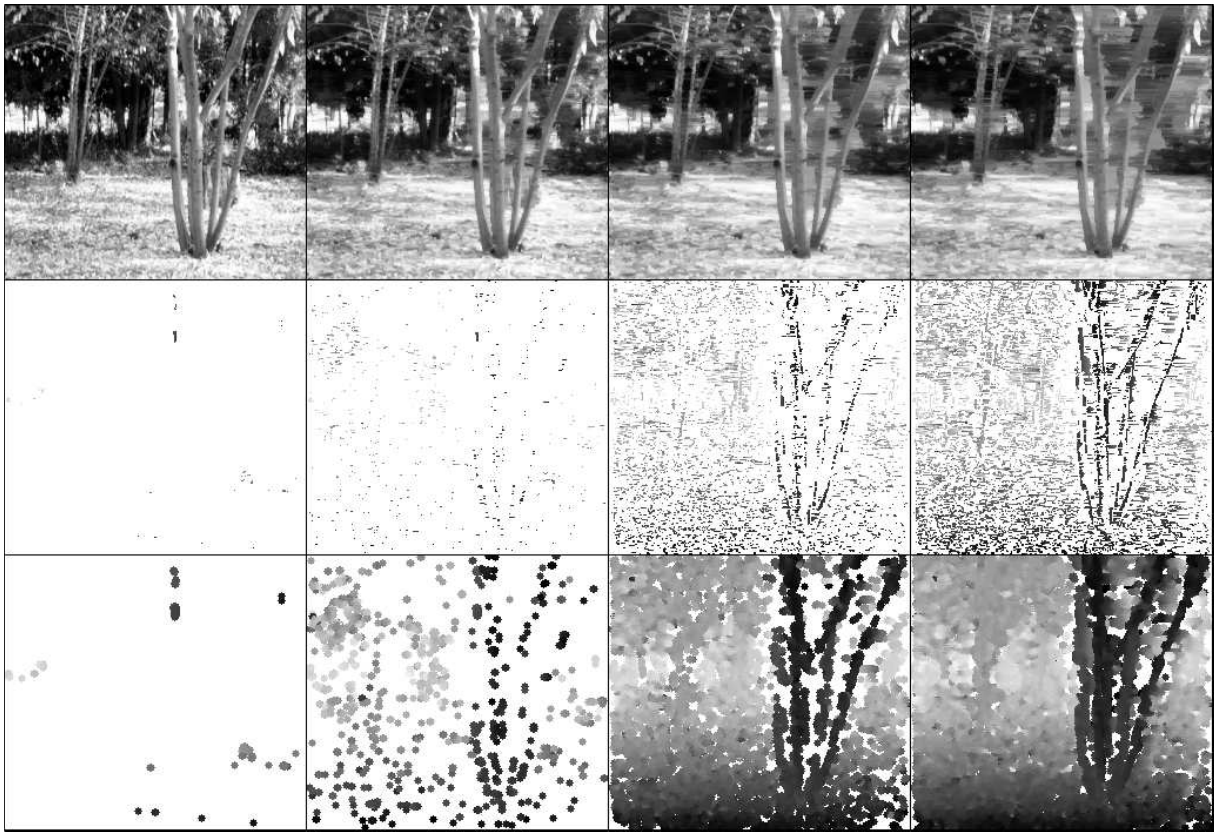

Figure 13.6: Computation of the velocity field (minimum of 15 matchings).

*The four images on the first row are taken from four different movies : each image is the 20th image (over 64) of the movie it belongs to. These movies result from the DCMA at different scales :*

| | |
|---|---|
| *column 1:* | *original "TREES" movie* |
| *column 2:* | *processed movie (5 iterations)* |
| *column 3:* | *processed movie (15 iterations)* |
| *column 4:* | *processed movie (31 iterations)* |

*Then, the velocity field of each movie was computed on the 20th image using the algorithm we described previously, with a matching constraint of 15 images. These velocities are represented on row 2 : the white color means points where no matching was found with respect to the constraint, and the grey scale (from light grey to black) measures the velocity from 0.0 to 2.0 pixels per image. On the third row, the velocity images of row 2 were "dilated" to produce more readable results. Notice how the velocity information, which is almost inexistant on the original movie (for the matching constraint we imposed), progressively appears on the DCMA as the scale increases. Since the distance of objects to the image plane is inversely proportional to their velocity, closest points appear in black and farthest ones in light grey. On the last image of row 3, we distinguish the foreground tree in black, the ground from black to middle grey, the background tree in middle grey, and the far background in light grey.*

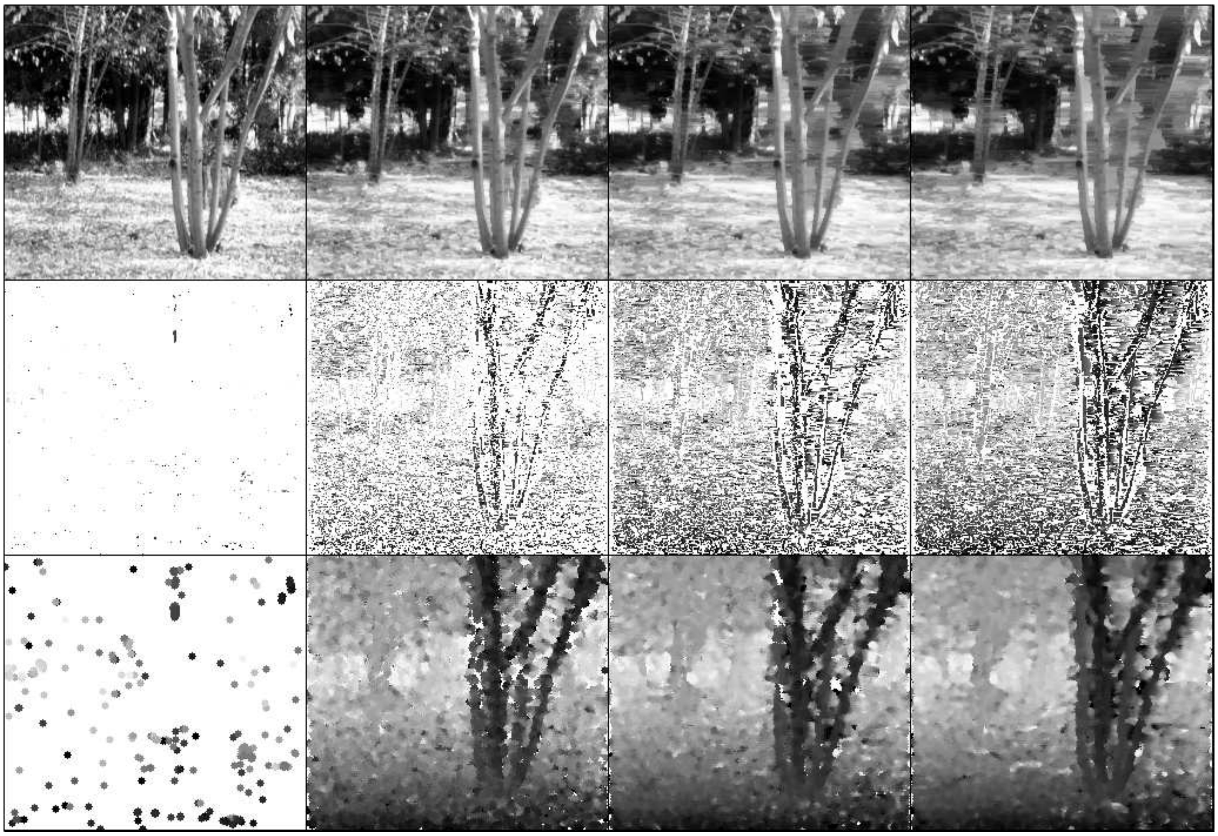

Figure 13.7: Computation of the velocity field (minimum of 5 matchings).

*The representation is the same as for Figure 13.6, but this time, the velocities were computed with a less restrictive matching constraint of 5 images (instead of 15 for Figure 13.6).*

*The velocity images we obtain (row 2) are more dense because new computable velocities appear. However, these new obtained velocities are less reliable due to the less restrictive matching constraint. This explains the noisy appearence of the images on row 3 compared to those of Figure 13.6. Notice that this noise decreases as the scale of analysis increases : this is coherent with the theory which predicts that the velocity field is progressively smoothed as the scale of analysis increases (see Proposition 37).*