# A Depth-Compatible Multiscale Analysis of Movies

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**Abstract**

Using an axiomatic formulation, we devise a diffusion equation on movies that preserves the 3D structure of objects, an interesting property for depth recovery tasks. Since this equation presents a strong singularity and is not handled by the classical theory of viscosity solutions, we study the existence and unicity of weak solutions, and point out some properties, including a variational interpretation. Last, we propose a very simple numerical scheme based on inf-sup operators and show some experiments on a real movie.

**Key words :** second order parabolic partial differential equation, non-linear multiscale analysis, image processing, structure from motion, projective invariance, variational problem, numerical scheme.

## 1 Introduction : the depth recovery problem

A recurrent aim of computer vision is to recover the tridimensional structure of an object represented on images from different points of view. Theoretically, two views (stereovision) are sufficient, but in practice more information is required to ensure both a precise depth estimation and a reliable matching algorithm : this is the reason why people have focused their attention to the $n$-views case, defining the so-called structure-from-motion problem. Even when the scene is fixed and when the camera motion is totally or partially known, it is still a problem to take advantage of the depth redundancy contained in the movie.

Suppose that we observe a fixed surface $\Sigma$ represented by the graph of a depth function $Z(X, Y)$ with a camera moving along the $X$ axis, the focal plane remaining orthogonal to the $Z$ axis. Then, calling $C(\theta)$ the abcissa of the camera at time $\theta$, each point $M(X, Y, Z(X, Y)$ of $\Sigma$ is projected on the image plane $Z = a$ (a being the focal length) into a point $P(x, y)$ such that

$$x = a\frac{X - C(\theta)}{Z(X, Y)} \qquad \text{and} \qquad y = a\frac{Y}{Z(X, Y)} \tag{1}$$

In the following, we shall make an important restriction to admissible surface/camera motion pairs : we suppose that no occlusion arise, that is to say that the projection

$M \mapsto P$ is injective at any time $\theta$. Then, if $\Sigma$ is a Lambertian surface characterized by its luminance $U(X,Y)$, the camera should produce an ideal movie $u : \mathbb{R}^3 \to \mathbb{R}$ such that

$$u\left(a\frac{X - C(\theta)}{Z(X,Y)}, a\frac{Y}{Z(X,Y)}, \theta\right) = g \circ U(X,Y), \tag{2}$$

$g$ being a contrast change (a nondecreasing scalar map) depending on the camera. The aim of structure from motion is to determine the depth map $Z(X,Y)$ from the observation of the movie $u(x,y,\theta)$. The classical way is to derive (2) with respect with $\theta$, which vanishes the total derivative

$$\frac{\mathrm{D}u}{\mathrm{D}\theta} = u_\theta + v u_x = 0, \tag{3}$$

the velocity field $v$ being given by

$$v = -a\frac{C'(\theta)}{Z(X,Y)}. \tag{4}$$

This identification of $v$ between (3) and (4) should permit to recover at any time the depth $Z$ up to a multiplicative constant $aC'(\theta)$ at each point where $u_\theta/u_x$ exists. This is a difficult problem since $u_\theta/u_x$ is not easy to compute (and often even not defined) ; moreover, real movies are not exactly ideal in practice, so that each choice of $\theta$ may produce a different estimation of $Z$. We study in the next section how to define multiscale analyses which "idealize" movies.

# 2 An axiomatic formulation

Let us first define some notations. Given an open or closed subset $\Omega$ of $\mathbb{R}^n$, $C^n(\Omega)$ means the space of continuous maps $u : \overline{\Omega} \to \mathbb{R}$ of class $C^n$ on $\Omega$. As usual, $\overline{\Omega}$ means the topological closure of $\Omega$ in $\mathbb{R}^n$. We shall also write $\mathcal{S}(\mathbb{R}^3)$ to denote the set of real symmetric 3x3 matrices. As we saw previously, a movie is a real-valued map $u$ defined on a subset of $\mathbb{R}^3$, the value $u(x,y,\theta)$ representing the light intensity at a point $(x,y)$ of the plane at time $\theta$. The natural domain for a digital movie is $[x_1,x_2] \times [y_1,y_2] \times [\theta_1,\theta_2]$, but we shall see that it is simpler and more logical to suppose that a movie is defined on $\mathbb{R}^2 \times \overline{I}$, with either $I = ]\theta_1,\theta_2[$ or $I = S^1$ (case of a time-periodic movie). We recall that a multiscale analysis is a family of operators $(T_t : \mathcal{M} \to \mathcal{M})_{t \geqslant 0}$, $t$ representing the scale of analysis. Here, $\mathcal{M}$ is a movie space, that is to say a space of continuous real-valued maps defined on $\mathbb{R}^2 \times \overline{I}$. The choice of $\mathcal{M}$ will become natural later, but is not necessary for the time being since we only want to find constraints on $(T_t)$. However, because of the singularity which appears in the computation of the velocity field when the partial derivative $u_x$ vanishes, we shall suppose in the following that for any $n \geqslant 1$, the space

$$\mathcal{M}^n = \{u \in \mathcal{M} \cap C^n(\mathbb{R}^2 \times \overline{I}, \mathbb{R});\ \forall \mathbf{z} \in \mathbb{R}^3, u_x(\mathbf{z}) \neq 0\}$$

is nonempty, and that given $(\lambda, \mathbf{p}, A) \in \mathbb{R} \times \mathbb{R}^3 \times \mathcal{S}(\mathbb{R}^3)$, it is possible to find $u \in \mathcal{M}^2$ such that $u(\mathbf{0}) = \lambda$, $Du(\mathbf{0}) = \mathbf{p}$ and $D^2u(\mathbf{0}) = A$ (in all the following, $Du$ and $D^2u$ represent the spatiotemporal first and second derivatives of $u$, that is, the scale variable $t$ is not involved).

## 2.1 Architectural axioms

In the spirit of [1], we first constrain the multiscale analysis to satisfy three classical architectural axioms :

- **[Semi-group]** : $T_0 = Id$ and $\forall t, t' > 0, \quad T_{t+t'} = T_{t'} \circ T_t$.

- **[Local Comparison Principle]** :

if $u < \tilde{u}$ on a neighborhood of $\mathbf{z}$, then $T_t u(\mathbf{z}) \leqslant T_t \tilde{u}(\mathbf{z})$ for $t > 0$ small enough.

- **[Regularity]** : if $u$ is a quadratic form (that is, $u(\mathbf{z}) = [A](\mathbf{z}, \mathbf{z}) + \langle \mathbf{p}, \mathbf{z} \rangle + \lambda$ where $[A]$ is the bilinear map associated to the symmetric 3x3 matrix $A$, $\mathbf{p}$ a 3-dimensional vector whose first component $p_1$ is nonzero, and $\lambda$ a given constant), then

$$\lim_{t \to 0} \frac{T_t u - u}{t}(\mathbf{z}) = F(A, \mathbf{p}, \lambda)$$

and $F$ depends continuously on $A$.

These axioms can be found in the axiomatic characterization of the affine morphological scale space for example ; only the **[Regularity]** axiom has been adapted to the depth recovery problem. Please refer to [1] for complete discussion. The classification starts with the following theorem.

**Theorem 1** *A multiscale analysis $T_t : u_0(\cdot) \mapsto u(\cdot, t)$ satisfying* **[Semi-group]**, **[Local Comparison Principle]** *and* **[Regularity]** *can be described by a partial differential equation of the kind*

$$\frac{\partial u}{\partial t} = F(D^2 u, Du, u) \tag{5}$$

*submitted to initial condition $u(\cdot, 0) = u_0$. Moreover, $F$ is elliptic (that is, nondecreasing with respect to its first argument for the usual order on 3x3 symmetric matrices), and continuous with respect to its first argument.*

The proof of an equivalent theorem can be found in [1]. The existence of $F$ is a direct consequence of the **[Regularity]** axiom. The fact that the evolution is given by a PDE of order two (and not more) results from the **[Local Comparison Principle]** axiom, as well as the ellipticity of $F$. Notice that (5) makes sense (in terms of existence and unicity of solutions) according to the theory of viscosity solutions (see [4]), provided that the singularity $u_x = 0$ is not involved. This point will become clearer in the next section.

## 2.2 Specific axioms

We now come to specific axioms with respect to the depth recovery problem. First, remember that when $u \in \mathcal{M}^n$ ($n \geqslant 1$), the apparent velocity field operator is well defined by

$$v[u] := -\frac{u_\theta}{u_x}.$$

Since we are interested in the apparent velocity field, it seems natural that our analysis focuses mainly on this datum. In that sense, it is rather natural to constrain the analysis to commute with operators that preserve the apparent velocity field. This justifies the following axiom.

- **[v-Compatibility]**: For any $h : \mathbb{R}^4 \mapsto \mathbb{R}$ such that

$$\forall u \in \mathcal{M}^1,\ R_h u \in \mathcal{M}^1 \text{ and } v[R_h u] = v[u], \text{ with } R_h u(x,y,\theta) = h\left(u(x,y,\theta), x, y, \theta\right),$$

$$\text{we have} \qquad \forall t,\ T_t \circ R_h = R_h \circ T_t.$$

This axiom implies two weaker axioms, obtained for specific choices of $h$.

$\diamond$ **[Strong Morphological Invariance]**: For any monotone scalar map $g$,

$$\forall u \in \mathcal{M}^1, \forall t,\ T_t g(u) = g(T_t u).$$

$\diamond$ **[Transversal Invariance]**: For any nonvanishing scalar map $g$,

$$\forall u \in \mathcal{M}^1, \forall t,\ T_t(g(y) \cdot u) = g(y) \cdot (T_t u).$$

Notice that we implicitely supposed that $\mathcal{M}^1$ is stable under the operations $u \mapsto g \circ u$ and $u \mapsto g(y) \cdot u$. Following [10], we also constrain the analysis to commute with the superimposition of any uniform movement of the camera.

- **[Galilean Invariance]**:

$$\forall \alpha \in \mathbb{R},\ \forall u \in \mathcal{M},\ \forall t,\ T_t(u \circ B_\alpha) = (T_t u) \circ B_\alpha, \quad \text{with} \quad B_\alpha(x,y,\theta) = (x - \alpha\theta, y, \theta).$$

Last, we would like the analysis not to depend on the focal length of the camera (the $a$ variable in §1.1). This can be translated into a commutation with spatial homothetic transformations.

- **[Zoom Invariance]**:

$$\forall \lambda \neq 0,\ \forall u \in \mathcal{M}, \forall t,\ T_t(u \circ H_\lambda) = (T_t u) \circ H_\lambda, \quad \text{with} \quad H_\lambda(x,y,\theta) = (\lambda x, \lambda y, \theta).$$

## 2.3 Fundamental equation

In this section, we establish a necessary gorm for depth-compatible multiscale analyses (as usual, up to a rescaling $t \mapsto \lambda t$).

**Theorem 2** *There exists at most one multiscale analysis of movies defined on $\mathcal{M}^2$ satisfying the architectural axioms plus* **[v-Compatibility]**, **[Galilean Invariance]** *and* **[Zoom Invariance]**. *It must be described by the partial differential equation*

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x} u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}. \tag{6}$$

For the time being, (6) is defined in the classical sense for $u(\cdot,t) \in \mathcal{M}^2$. In fact, we shall see in §3 how to define weak solutions of (6) that are not in $\mathcal{M}^2$ but only continuous. Notice incidentally that (6) can be rewritten into

$$u_t = u_{\xi\xi} \quad \text{with} \quad \xi = (-\frac{u_\theta}{u_x}, 0, 1) \quad \text{and} \quad u_{\xi\xi} := [D^2u](\xi,\xi),$$

which describess an anisotropic diffusion of $u$ along the movement direction. The diffusion term involves the apparent acceleration of the movie, defined by

$$\Gamma := \frac{\mathrm{D}v}{\mathrm{D}\theta} = v_\theta + v v_x = -\frac{u_{\xi\xi}}{u_x}.$$

**Lemma 1** *For any multiscale analysis satisfying the architectural axioms and* **[v-Compatibility]***, there exists a map* $F : \mathbb{R}^2 \mapsto \mathbb{R}$ *such that*

$$u_t = u_x F(\Gamma, v). \tag{7}$$

*Proof :*

Let us first make clear that the map $F$ we write here is not the map $F$ of (5) : we kept the same letter to simplify the notation. We are going to use the fact that the **[v-Compatibility]** axiom implies the simpler axioms **[Strong Morphological Invariance]** and **[Transversal Invariance]**, as we noticed before. Applying **[Strong Morphological Invariance]** for $g(u) = u + \lambda$ ($\lambda$ being an arbitrary constant) implies that $F$ cannot depend on $u$ in (5) (see [1] for the complete proof), so that we have

$$\frac{\partial u}{\partial t} = G(D^2u, Du) \tag{8}$$

Now, the **[Transversal Invariance]** axiom states that for any nonvanishing function $g$ of class $C^2$,

$$\forall u \in \mathcal{M}^2, \forall y, \quad G(D^2(g(y)\cdot u), D(g(y)\cdot u)) = G(D^2u, Du). \tag{9}$$

Let $A = [a_{ij}] \in \mathcal{S}(\mathbb{R}^3)$, $\lambda \in \mathbb{R}$ and $\mathbf{p} = (p_i) \in \mathbb{R}^3$ such that $p_1 \neq 0$ (the coordinates $x, y, \theta$ will be indexed by $1, 2, 3$ in the following). By hypothesis on $\mathcal{M}^2$, we can build a movie $u \in \mathcal{M}^2$ such that $u(\mathbf{0}) = \lambda$, $Du(\mathbf{0}) = \mathbf{p}$ and $D^2u(\mathbf{0}) = A$. Now, consider the vector $\mathbf{y} = {}^T(0\ 1\ 0)$, $Q_y = \mathbf{y} \otimes \mathbf{y}$ the projection matrix on the line $\mathbb{R}\mathbf{y}$, and $Q_{y^\perp} = I - Q_y$ the projection matrix on the $(x,\theta)$ plane. Applying (9) to $u$ in $\mathbf{0}$, we obtain

$$G(g(0)A + g'(0)\mathbf{y} \otimes \mathbf{p} + g''(0)\lambda Q_y, g(0)\mathbf{p} + g'(0)\mathbf{y}) = G(A, \mathbf{p}).$$

If we choose $g(y) = 1 + y^2/2$, we get

$$\forall A, \mathbf{p}, \lambda, \quad G(A + \lambda Q_y, \mathbf{p}) = G(A, \mathbf{p}), \tag{10}$$

and taking $\lambda = -a_{22}$ proves that $G$ does no depend on $a_{22}$.

Now we are going to show that $G$ does not depend on $a_{12}$ and $a_{23}$ either, thanks to the **[Local Comparison Principle]** axiom, by using a technique from Giga et Goto (see [9]). Let us define $A' = A - a_{22}Qy$ and for $\varepsilon > 0$,

$$I_\varepsilon = \varepsilon Q_{y^\perp} + \frac{a_{21}^2 + a_{23}^2}{\varepsilon} Q_y = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \dfrac{a_{21}^2 + a_{23}^2}{\varepsilon} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}.$$

The characteristic polynom of the matrix

$$A_\varepsilon = Q_{y^\perp} A' Q_{y^\perp} - A' + I_\varepsilon = \begin{pmatrix} \varepsilon & -a_{21} & 0 \\ -a_{21} & \dfrac{a_{21}^2 + a_{23}^2}{\varepsilon} & -a_{23} \\ 0 & -a_{23} & \varepsilon \end{pmatrix}$$

$$\text{is} \qquad \det(xI - A_\varepsilon) = x(x-\varepsilon)\left(x - (\varepsilon + \frac{a_{21}^2 + a_{23}^2}{\varepsilon})\right).$$

As the eigenvalues of $A_\varepsilon$ are nonnegative, $A_\varepsilon$ is positive (for the usual order in $\mathcal{S}(\mathbb{R}^3)$), and symmetrically $A_{-\varepsilon}$ is negative, which yields

$$A' - I_\varepsilon \leqslant Q_{y^\perp} A' Q_{y^\perp} \leqslant A' + I_\varepsilon$$

But the **[Local Comparison Principle]** axiom implies (see [9])

$$\forall A, B, \mathbf{p}, \quad A \geqslant B \ \Rightarrow\ G(A, p) \geqslant G(B, \mathbf{p}),$$

$$\text{so that} \quad \forall A, \mathbf{p}, \quad G(A' - I_\varepsilon, \mathbf{p}) \leqslant G(Q_{y^\perp} A' Q_{y^\perp}, \mathbf{p}) \leqslant G(A' + I_\varepsilon, \mathbf{p}).$$

Then, using (10), we get

$$\forall A, \mathbf{p}, \quad G(A + \varepsilon I, \mathbf{p}) \leqslant G(Q_{y^\perp} A Q_{y^\perp}, \mathbf{p}) \leqslant G(A + \varepsilon I, \mathbf{p})$$

and taking the limit when $\varepsilon \to 0$, the continuity of $G$ implies

$$\forall A, \mathbf{p}, \quad G(A, \mathbf{p}) = G(Q_{y^\perp} A Q_{y^\perp}, \mathbf{p}). \tag{11}$$

But remember (see [1]) that the **[Local Comparison Principle]** and **[Strong Morphological Invariance]** axioms also constrain

$$\forall A, \mathbf{p}, \quad G(A, \mathbf{p}) = G(Q_{\mathbf{p}^\perp} A Q_{\mathbf{p}^\perp}, \mathbf{p}).$$

In combination with (11), we obtain

$$\forall A, \mathbf{p}, \quad G(A, \mathbf{p}) = G(Q_{\mathbf{p}^\perp} Q_{y^\perp} A Q_{y^\perp} Q_{\mathbf{p}^\perp}, \mathbf{p}),$$

where $Q_{\mathbf{p}^\perp} Q_{y^\perp}$ is nothing but the projection on the line $(\mathbf{p})^\perp \cap (\mathbf{y})^\perp = \mathbb{R}\xi$, with $\xi = {}^t(-p_3/p_1, 0, 1)$. Hence, since $\Gamma = [D^2 u](\xi, \xi)$, we can write the evolution of $u$ under the form

$$u_t = F(\Gamma, u_x, u_y, u_\theta). \tag{12}$$

Now, applying the **[Transversal Invariance]** axiom to $F$, we obtain

$$\forall A, \mathbf{p}, g, y \quad F([A](\xi,\xi), p_1, p_2, p_3) = F([A](\xi,\xi), p_1, g'(y)p_2, p_3),$$

from which it is easy to deduce that $F$ does not depend on $p_2$. Hence, we can rewrite (12) into $u_t = u_x G(\Gamma, v, u_x)$, and apply the **[Strong Morphological Invariance]** axiom. Since $\Gamma$ and $v$ are morphological operators, it yields

$$\forall u,\ \forall \lambda \neq 0, \quad G(\Gamma, v, u_x) = G(\Gamma, v, \lambda u_x). \tag{13}$$

For any $(\alpha, \beta) \in \mathbb{R}^2$, we consider a movie $u \in \mathcal{M}^2$ such that, in a neighborhood of $\mathbf{0}$,

$$u(x,y,\theta) = \frac{\alpha}{2}\theta^2 + x - \beta\theta + o(\theta^2 + x^2).$$

We have $u_x(\mathbf{0}) = 1$, $v(\mathbf{0}) = \beta$ and $\Gamma(\mathbf{0}) = \alpha$ so that (13) can be rewritten into

$$\forall \alpha, \beta,\ \forall \lambda \neq 0,\ \ G(\alpha, \beta, 1) = G(\alpha, \beta, \lambda),$$

which means that $G$ does not depend on its third argument (notice that $G$ does not need to be defined when $u_x = 0$). Hence, the evolution of $u$ is given by (7). □

**Remark :** We proved that the **[v-Compatibility]** axiom, in association with the architectural axioms, implies that the evolution of $u$ does not involve the $y$ coordinate : in other words, the sliced images $(x,\theta) \mapsto u(x,y,\theta)$ (with $y$ fixed) are processed independently. In the following, we shall often ignore the $y$ coordinate and we shall write $u(x,\theta)$ instead of $u(x,y,\theta)$, the $y$ variable being supposed fixed.

**Lemma 2** *A multiscale analysis satisfying the architectural axioms plus* **[v-Compatibility]** *and* **[Galilean Invariance]** *can be written*

$$u_t = u_x F(\Gamma) \tag{14}$$

*Proof :*

Since the multiscale analysis commutes with $B_\alpha$, we have

$$\frac{\partial}{\partial t}(u \circ B_\alpha) = \frac{\partial u}{\partial t} \circ B_\alpha.$$

Writing $\tilde{u} = u \circ B_\alpha$ yields

$$\tilde{u}_x = \frac{\partial}{\partial x} u(x - \alpha\theta, \theta) = u_x \circ B_\alpha, \qquad \tilde{u}_\theta = \frac{\partial}{\partial \theta} u(x - \alpha\theta, \theta) = (u_\theta - \alpha u_x) \circ B_\alpha$$

$$\tilde{v} = -\frac{\tilde{u}_\theta}{\tilde{u}_x} = v \circ B_\alpha + \alpha \ \text{ and } \ \tilde{\Gamma} = \tilde{v}_\theta + \tilde{v}\tilde{v}_x = (v_\theta - \alpha v_x + (v + \alpha)v_x) \circ B_\alpha = \Gamma \circ B_\alpha.$$

From Lemma 1 we know that $u_t = u_x F(\Gamma, v)$. Hence,

$$\forall u, \alpha, \quad u_x F(\Gamma, v + \alpha) = u_x F(\Gamma, v),$$

so that $F$ does not depend on its second argument. □

**Lemma 3** *A multiscale analysis satisfying the architectural axioms plus* **[v-Compatibility]** *and* **[Zoom Invariance]** *can be written*

$$u_t = \begin{cases} u_\theta F(\dfrac{\Gamma}{v}) \text{ if } u_\theta \neq 0, \\ \\ a u_x \text{ if } u_\theta = 0. \end{cases} \tag{15}$$

*Proof :*

We proceed as for Lemma 2 : writing $\tilde{u} = u \circ H_\lambda$, we get

$$\begin{aligned} \tilde{v} &= -\frac{\tilde{u}_\theta}{\tilde{u}_x} = -\frac{u_\theta}{\lambda u_x} \circ H_\lambda = \frac{v}{\lambda} \circ H_\lambda \\ \text{and} \quad \tilde{\Gamma} &= \tilde{v}_\theta + \tilde{v}\tilde{v}_x = (\frac{v_\theta}{\lambda} + \frac{v}{\lambda}\frac{\lambda v_x}{\lambda}) \circ H_\lambda = \frac{\Gamma}{\lambda} \circ H_\lambda. \end{aligned}$$

We can write (7) as

$$u_t = u_x F(\Gamma, v) = u_\theta G(\frac{\Gamma}{v}, v)$$

everywhere $u_\theta \neq 0$, and since the evolution commutes with $H_\lambda$, we have

$$\forall u, \lambda, \quad u_\theta G(\frac{\Gamma}{v}, v) = u_\theta G(\frac{\Gamma}{v}, \frac{v}{\lambda}).$$

Taking the limit $\lambda \to \infty$ proves that $G$ cannot depend on its second argument. Besides, everywhere $u_\theta = 0$ we have

$$\forall u, \lambda, \quad u_x F(\Gamma, 0) = u_x F(\frac{\Gamma}{\lambda}, 0),$$

so that $F(\Gamma, 0) = F(0, 0)$. □

*Proof of Theorem 2 :*

If a multiscale analysis satisfies the axioms of Theorem 2, the corresponding evolution equation can be written in both forms given in (14) and (15). But the only common case is, up to the usual rescaling $t \mapsto \lambda t$,

$$u_t = -u_x \Gamma = u_\theta \frac{\Gamma}{v} = u_{\xi\xi},$$

which is the announced equation. □

Conversely, we have to check that it is possible to define from (6) a multiscale analysis of movies satisfying the previous axioms. This is the aim of §3.

# 3 The Depth-Compatible Multiscale Analysis

In this section, we give a rigorous sense to the DCMA evolution induced by (6)[1].

## 3.1 Classical solutions of the DCMA

For the reason we explained before, we forget the $y$ variable in the following, and a movie is defined on $\mathbb{R}\times\overline{I}$, with either $I=]\theta_1,\theta_2[$ or $I=S^1$. In the space variable, a periodization has no meaning in terms of scene interpretation, so that we shall rather suppose that $u$ tends towards some constant when $x$ grows to infinity (we shall say that $u$ is "constant at infinity"). Notice that such a condition is classical, even in a more restrictive formulation (e.g. $u$ *equals* a constant outside a compact set, see [5] for example).

**Definition 1** *For $c=(c^-,c^+)\in\mathbb{R}^2$ and $n\geqslant 0$, $\mathcal{C}_c^n$ is the space of movies $u\in C^n(\mathbb{R}\times I)$ such that*

$$\sup_{\theta\in\overline{I}}|u(-x,\theta)-c^-|+|u(x,\theta)-c^+|\to 0 \quad \text{as} \quad x\to+\infty. \tag{16}$$

In all the following, we write $\Omega=\mathbb{R}\times I\times]0,+\infty[$, that is, $\overline{\Omega}$ is the domain of movie analyses.

**Definition 2** *For $c\in\mathbb{R}^2$ and $n,p\geqslant 0$, $\mathcal{C}_c^{n,p}$ is the space of movie analyses $u\in C^0(\overline{\Omega})$ such that*

1. $\forall T>0, \quad \sup_{\theta\in\overline{I},t\leqslant T}|u(-x,\theta,t)-c^-|+|u(x,\theta,t)-c^+|\to 0 \quad \text{as} \quad x\to+\infty,$
2. *on $\Omega$, $(x,\theta,t)\mapsto u(x,\theta,t)$ is of class $C^n$ with respect to $(x,\theta)$ and $C^p$ with respect to $t$.*

Now we want to define classical solutions of (6). However, the space $\mathcal{M}^2$ we introduced in the axiomatic formulation is too restrictive, because of the condition $u_x\neq 0$. This is the reason why we shall write a degenerate formulation of (6) when $u_x$ vanishes. For example, let us onsider $g\in C^2(\mathbb{R})$ such that $g(x)\to 0$ as $|x|\to+\infty$. We can define the movie analysis $u:\mathbb{R}\times S^1\times[0,+\infty[\to\mathbb{R}$ by

$$u(x,\theta,t)=g(x-\theta^2-2t),$$

the representant of $\theta$ being taken in $[-\pi,\pi[$. Then, (6) is satisfied by $u$ at any point where $u_x\neq 0$, and when $u_x=0$ we have also $u_t=0$. This suggests a simple degenerate formulation of (6) when $u_x$ vanishes.

[1]The reason why we call this evolution equation DCMA (for Depth-Compatible Multiscale Analysis) will become clear in §4.

**Definition 3** *Given $u_0 \in \mathcal{C}^0_c$, we say that $u$ is a classical solution of the DCMA associated to the initial datum $u_0$ if*

*(i)* $u \in \mathcal{C}^{2,1}_c$,

*(ii) on* $\Omega = \mathbb{R} \times I \times ]0, +\infty[$,

$$\begin{cases} u_t = u_{\theta\theta} - 2\dfrac{u_\theta}{u_x}u_{\theta x} + (\dfrac{u_\theta}{u_x})^2 u_{xx} & \text{when} \quad u_x \neq 0, \\ \\ u_t = 0 \qquad \text{when} \quad u_x = 0. \end{cases}$$

*(iii)* $\forall (x, \theta, t) \in \partial\Omega,\ \ u(x, \theta, t) = u_0(x, \theta)$.

In order to establish a uniqueness result, we first prove a comparison principle.

**Lemma 4 (comparison principle)** *Let $u$ and $\tilde{u}$ be classical solutions of the DCMA associated to initial data $u_0$ and $\tilde{u}_0$ respectively. If $u_0 \leqslant \tilde{u}_0$, then $u \leqslant \tilde{u}$ on $\Omega$.*

*Proof :*
For $R > 0$ and $T > 0$, let us write

$$\varepsilon(R, T) = \sup_{|x| \geqslant R, \theta \in \overline{I}, t \leqslant T} u(x, \theta, t) - \tilde{u}(x, \theta, t).$$

Since $u$ and $\tilde{u}$ belong to $\mathcal{C}^{2,1}_c$ and $\mathcal{C}^{2,1}_{\tilde{c}}$, we have

$$\varepsilon(R, T) \to \varepsilon_0 = \max(c^- - \tilde{c}^-, c^+ - \tilde{c}^+) \quad \text{as} \quad R \to +\infty,$$

with $\varepsilon_0 \leqslant 0$ because $u_0 \leqslant \tilde{u}_0$. For $\alpha > 0$, we consider

$$\Lambda(x, \theta, t) = u(x, \theta, t) - \tilde{u}(x, \theta, t) - \alpha t.$$

On the compact set $K_{R,T} = [-R, R] \times \overline{I} \times [0, T]$, the continuous map $\Lambda$ attains its maximum value at a point $\mathbf{z}_0 = (x_0, \theta_0, t_0)$.

1. Suppose that

$$|x_0| < R, \quad \theta_0 \in I \quad \text{and} \quad t_0 \in ]0, T]. \tag{17}$$

In $\mathbf{z}_0$ we have $\Lambda_x = \Lambda_\theta = 0$, $\Lambda_t \geqslant 0$ and $D^2\Lambda \leqslant 0$, which yields

$$Du(\mathbf{z}_0) = D\tilde{u}(\mathbf{z}_0), \tag{18}$$

$$u_t(\mathbf{z}_0) - \tilde{u}_t(\mathbf{z}_0) \geqslant \alpha, \tag{19}$$

$$\text{and} \quad D^2 u(\mathbf{z}_0) \leqslant D^2 \tilde{u}(\mathbf{z}_0), \tag{20}$$

the last inequality being meant for the usual order on symmetric 2x2 matrices.

1.a. If $u_x(\mathbf{z}_0) \neq 0$, then $\tilde{u}_x(\mathbf{z}_0) = u_x(\mathbf{z}_0) \neq 0$. Now recall that

$$u_t = u_{\theta\theta} - 2\frac{u_\theta}{u_x} u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx} = F(D^2 u, Du),$$

where $F$ is an elliptic operator, that is to say nondecreasing with respect to its first argument. Hence, (18) and (20) imply $u_t(\mathbf{z}_0) \leqslant \tilde{u}_t(\mathbf{z}_0)$, which contradicts (19).

1.b. If $u_x(\mathbf{z}_0) = 0$, then $\tilde{u}_x(\mathbf{z}_0) = 0$, and since $u$ and $\tilde{u}$ are solutions of the DCMA, we have $u_t(z_0) = \tilde{u}_t(z_0) = 0$, which is a contradiction with (19).

2. As a consequence of 1.a and 1.b, (17) is false and necessarily we have either $|x_0| = R$ or $\theta_0 \in \partial I$ or $t_0 = 0$. If $|x_0| = R$, then $\Lambda(x_0, \theta_0, t_0) \leqslant \varepsilon(R, T)$, while $\Lambda(x_0, \theta_0, t_0) \leqslant 0$ when $\theta_0 \in \partial I$ or $t_0 = 0$. Consequently, we have

$$\max_{K_{R,T}} u \leqslant \tilde{u} \leqslant \max(0, \varepsilon(R, T)) + \alpha T,$$

and making $\alpha \to 0$ proves that

$$u \leqslant \tilde{u} + \max(0, \varepsilon(R, T)) \qquad \text{on} \quad [-R, R] \times \overline{I} \times [0, T].$$

Last, sending $R$ to infinity forces $\max(0, \varepsilon(R, T))$ to vanish so that $u \leqslant \tilde{u}$ on $\mathbb{R}\overline{I} \times [0, T]$ for any value of $T$. □

**Corollary 1 (contraction property)** *If $u$ and $\tilde{u}$ are two classical solutions of the DCMA associated to the initial data $u_0$ and $\tilde{u}_0$, then*

$$\|u - \tilde{u}\|_\infty \leqslant \|u_0 - \tilde{u}_0\|_\infty.$$

**Corollary 2 (uniqueness)** *A classical solution of the DCMA associated to a given initial datum $u_0 \in \mathcal{C}_c^2$ is unique.*

In order to ensure the existence of classical solutions of the DCMA, we now restrain the space of initial data.

**Definition 4** *For $n \geqslant 1$, we write $\mathcal{V}_c^n$ the space of movies $u \in \mathcal{C}_c^n$ for which there exists a movie $v \in \mathcal{C}_0^{n-1}$ such that*

$$u_\theta + v u_x = 0 \quad \text{on} \quad \mathbb{R} \times \overline{I}. \tag{21}$$

*$v$ is called a velocity map of $u$. In addition, the space $\mathcal{V}_c^{n,p}$ is defined as elements of $\mathcal{C}_c^{n,p}$ admitting a velocity map $v \in \mathcal{C}_0^{n-1,p}$.*

**Remark :** Consider a movie $u \in \mathcal{V}_c^n$. If $u_x(x, \theta) \neq 0$, $v(x, \theta)$ is uniquely determined because (21) forces

$$v(x, \theta) = -\frac{u_\theta}{u_x}(x, \theta).$$

When $u_x(x,\theta)=0$, (21) implies $u_\theta(x,\theta)=0$, and if $n\geqslant 2$, differentiating (21) with respect to $\theta$ and $x$ shows that $u_{\theta\theta}+2vu_{\theta x}+v^2u_{xx}=0$ as soon as $u_x=0$. A consequence is that if $u\in\mathcal{V}_c^{2,1}$ is a classical solution of the DCMA, then any velocity map $v$ of $u$ satisfies on $\Omega$

$$\left\{\begin{array}{l} u_\theta+vu_x=0 \\ u_t=u_{\theta\theta}+vu_{\theta x}+v^2u_{xx}. \end{array}\right. \tag{22}$$

**Proposition 1 (existence)** *Given an initial datum $u_0\in\mathcal{V}_c^n$ ($n\geqslant 2$), there exists a unique classical solution of the DCMA, and it belongs to $\mathcal{V}_c^{n,n}$.*

*Proof :*

The existence will be a consequence of Lemma 6 (which follows), and the uniqueness follows from Corollary 2. □

We are going to build explicit solutions of the DCMA. The idea is to notice that the trajectories (i.e. the curves $x(\theta)$ along which $u$ is constant) are smoothed by the monodimensional heat equation. For that purpose, we need to introduce the natural domain $I^\star$ for such trajectories. If $I=]\theta_1,\theta_2[$ then $I^\star=I$, and if $I=S^1$, then $I^\star=\mathbb{R}$ (the natural injection $S^1\hookrightarrow[0,2\pi[\subset\mathbb{R}$ being implicit). To simplify the notations, we suppose in the following that $0\in\overline{I}$.

**Definition 5** *A map $\varphi\in C^n(\mathbb{R}\times I^\star)$ ($n\geqslant 0$) is a* **$\theta$-graph** *of $u\in\mathcal{C}_c^n$ if*

1. *for any $\theta\in\overline{I^\star}$, the map $x\mapsto\varphi(x,\theta)$ is increasing and bijective (and $\varphi_x$ does not vanish if $n\geqslant 1$).*
2. *for any $(x,\theta)\in\mathbb{R}\times\overline{I^\star}$,*
$$u(\varphi(x,\theta),\theta)=u(x,0), \tag{23}$$
3. *for any $x\in\mathbb{R}$, $\varphi(x,0)=x$, and if $I=S^1$, then for any $(x,\theta)\in\mathbb{R}\times\overline{I^\star}$,*
$$\varphi(x,\theta+2\pi)=\varphi(\varphi(x,2\pi),\theta), \tag{24}$$
4. $$\sup_{|x|\geqslant R,\theta\in\overline{I}}|\varphi_\theta(x,\theta)|\to 0 \quad \text{as} \quad R\to+\infty$$ *(in a generalized sense if $n=0$).*

**Remark :** Notice that in Condition 4, the sup is taken for $\theta\in\overline{I}$ and not for $\theta\in\overline{I^\star}$. If $n=0$, the term $|\varphi_\theta(x,\theta)|$ must be replaced by

$$\limsup_{h\to 0}|\frac{\varphi(x,\theta+h)-\varphi(x,\theta)}{h}|.$$

**Lemma 5** *A movie $u\in\mathcal{C}_c^n$ ($n\geqslant 2$) belongs to $\mathcal{V}_c^n$ if and only if it admits a $\theta$-graph of class $C^n$.*

*Proof :*

1. Suppose that $u$ admits a $\theta$-graph of class $C^n$. Then, Condition 1 implies that the relation

$$v(\varphi(x,\theta),\theta) = \varphi_\theta(x,\theta) \tag{25}$$

defines a unique map $v$ of class $C^{n-1}$ on $\mathbb{R}\times\overline{I}$ (if $I = S^1$, (24) ensures the periodicity of $v$ in the $\theta$ variable). Then, differentiating (23) with respect to $\theta$ yields

$$\forall (x,\theta)\in\mathbb{R}\times\overline{I},\quad \varphi_\theta(x,\theta)u_x(\varphi(x,\theta),\theta) + u_\theta(\varphi(x,\theta),\theta) = 0,$$

so that $v$ is a velocity map of $u$ thanks to (25).

Now let us write $\operatorname{diam}(I)$ the diameter of $I$. Given $\varepsilon > 0$, Condition 4 ensures the existence of a $R > 0$ such that

$$\forall (x,\theta)\in\mathbb{R}\times\overline{I},\qquad |x|\geqslant R \;\Rightarrow\; |\varphi_\theta(x,\theta)|\leqslant\varepsilon.$$

Hence, if $|x|\geqslant R' = R+\varepsilon\cdot\operatorname{diam}(I)$ we have

$$\varphi(x,\theta) = \varphi(x,0) + \int_0^\theta \varphi_\theta(x,\tau)d\tau \geqslant x - \varepsilon|\theta| \geqslant R$$

and consequently

$$\sup_{|x|\geqslant R',\theta\in\overline{I}} |v(x,\theta)|\leqslant\varepsilon.$$

It follows that $v\in\mathcal{C}_0^{n-1}$ and the same reasoning proves that $u$ is constant at infinity, so that $u\in\mathcal{V}_c^n$.

2. Conversely, if $u\in\mathcal{V}_c^n$, consider a velocity movie $v$ of $u$. Given $(x_0,\theta_0)\in\mathbb{R}\times\overline{I}$, there exists a unique solution $X\in C^n(\overline{I^\star})$ of the ordinary differential equation

$$\frac{dX}{d\theta}(\theta) = v(X(\theta),\theta) \tag{26}$$

submitted to the condition $X(\theta_0) = x_0$. Since $v\in\mathcal{C}_0^{n-1}$, $v$ is bounded, so that $X$ is defined on the whole interval $\overline{I^\star}$. Call $\varphi(x_0,\theta)$ the solution $X$ associated to $\theta_0 = 0$, and let $k = \operatorname{diam}(I)\cdot\|v\|_\infty$. Then

$$\sup_{|x|\geqslant R,\theta\in\overline{I}} |\varphi_\theta(x,\theta)| \leqslant \sup_{|x|\geqslant R-k,\theta\in\overline{I}} |v(x,\theta)| \to 0 \quad\text{as}\quad R\to+\infty,$$

so that Condition 4 is satisfied for $\varphi$. In addition, since

$$\frac{\partial}{\partial\theta}(\varphi_x)(x,\theta) = \varphi_x(x,\theta)v_x(\varphi(x,\theta),\theta)$$

and $\varphi_x(x,0) = 1$, we have

$$\varphi_x(x,\theta) = \exp\left(\int_0^\theta v_x(\varphi(x,\Theta),\Theta)d\Theta\right),$$

so that $\varphi_x$ never vanishes. Now, suppose that the value $x_0$ is not attained by the map $x \mapsto \varphi(x, \theta_0)$ for a given value $\theta_0$. By considering the ODE (26) submitted to initial condition $X(\theta_0) = x_0$, we obtain the existence of a value $X(0)$ such that $\varphi(X(0), \theta_0) = x_0$, which is a contradiction. Hence, the map $x \mapsto \varphi(x, \theta_0)$ is surjective and Condition 1 is satisfied. If $I = S^1$, (24) is satisfied by $\varphi$ simply because $v$ is $2\pi$-periodic in the $\theta$ variable.

Last, a classical theorem (dependency with initial conditions) states that $\varphi$ is $C^n$ and we can write

$$\frac{d}{d\theta}\left(u(\varphi(x,\theta),\theta)\right) = \varphi_\theta(x,\theta)u_x(\varphi(x,\theta),\theta) + u_\theta(\varphi(x,\theta),\theta) = (vu_x + u_\theta)\left(\varphi(x,\theta),\theta\right) = 0.$$

This equation can be integrated to yield, for any $(x, \theta) \in \mathbb{R} \times \overline{I}$,

$$u(\varphi(x,\theta),\theta) = u(\varphi(x,0),0) = u(x,0),$$

so that Condition 2 is satisfied and $\varphi$ is a $\theta$-graph of $u$ of class $C^n$. $\square$

**Lemma 6** *Let $u_0 \in \mathcal{V}_c^n$ $(n \geqslant 2)$, and $\varphi_0$ be a $\theta$-graph of $u_0$ of class $C^n$. Define $(x, \theta, t) \mapsto \varphi(x, \theta, t)$ as the unique solution of the monodimensional heat equation*

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial \theta^2} \tag{27}$$

*on $\Omega^\star = \mathbb{R} \times I^\star \times ]0, +\infty[$ submitted to the boundary condition*

$$\forall (x,\theta,t) \in \partial\Omega^\star, \;\; \varphi(x,\theta,t) = \varphi_0(x,\theta). \tag{28}$$

*Then, the unique map $u : \overline{\Omega} \to \mathbb{R}$ defined by*

$$\forall (x,\theta,t) \in \overline{\Omega}, \;\; u(\varphi(x,\theta,t),\theta,t) = u_0(x,0) \tag{29}$$

*belongs to $\mathcal{V}_c^{n,n}$ and is a classical solution of the DCMA associated to the initial datum $u_0$.*

*Proof :*

1. By linearity, $\varphi_x$ is also solution of the heat equation. Hence, since the heat equation satisfies the comparison principle, the condition

$$\forall (x,\theta) \in \mathbb{R} \times \overline{I^\star}, \qquad (\varphi_0)_x(x,\theta) > 0$$

is preserved along evolution so that

$$\forall (x,\theta,t) \in \mathbb{R} \times \overline{I^\star} \times [0, +\infty[, \qquad \varphi_x(x,\theta,t) > 0.$$

2. Now we prove that $x \mapsto \varphi(x, \theta, t)$ is surjective. Condition 4 of Definition 5 shows that we can find two constants $A$ and $B$ (with $B = 0$ if $I^\star$ is bounded) such that

$$|\varphi_0(x,\theta) - x| \leqslant A + B|\theta|$$

on $\mathbb{R} \times \overline{I}$. If $I = S^1$, (24) extends this property to $\mathbb{R} \times \overline{I^\star}$. Evaluating the evolution of $\theta \mapsto |\theta|$ under the monodimensional heat Equation, we deduce that

$$\forall (x,\theta,t) \in \overline{\Omega^\star}, \quad |\varphi(x,\theta,t) - x| \leqslant A + B|\theta| + B\sqrt{\frac{4t}{\pi}}. \tag{30}$$

As a consequence, for any $(\theta,t) \in \overline{I} \times ]0,+\infty[$, $x \mapsto \varphi(x,\theta,t)$ is surjective.

3. Hence, (29) defines a unique map $u : \overline{\Omega} \to \mathbb{R}$ and a proof similar to the one of Lemma 5 shows that $u \in \mathcal{V}_c^{n,n}$ thanks to (30).

4. As concerns the boundary condition, it follows from (28), (29) and from the bijectivity of the map

$$\begin{array}{ccc} \partial\Omega & \to & \partial\Omega \\ (x,\theta,t) & \mapsto & (\varphi_0(x,\theta),\theta,t). \end{array}$$

5. Last, we prove that condition (ii) of Definition 3 is satisfied by $u$. Let us note $\mathbf{z}_1 = (\varphi(\mathbf{z}),\theta,t)$ for a given $\mathbf{z} \in \Omega$. If $u_x(\mathbf{z}_1) = 0$, differentiating (29) with respect to $t$ yields

$$\varphi_t(\mathbf{z})u_x(\mathbf{z}_1) + u_t(\mathbf{z}_1) = u_t(\mathbf{z}_1) = 0$$

as expected. If $u_x(\mathbf{z}_1) \neq 0$, we obtain $u_t(\mathbf{z}_1) = -\varphi_t(\mathbf{z})u_x(\mathbf{z}_1)$,

$$\frac{d}{d\theta}(u_0(x,0)) = 0 = \varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t), \qquad \text{and}$$

$$\begin{aligned} \frac{d^2}{d\theta^2}(u_0(x,0)) = 0 &= \frac{d}{d\theta}\left(\varphi_\theta(\mathbf{z})u_x(\varphi(\mathbf{z}),\theta,t) + u_\theta(\varphi(\mathbf{z}),\theta,t)\right) \\ &= \varphi_{\theta\theta}(\mathbf{z})u_x(\mathbf{z}_1) + \varphi_\theta^2(\mathbf{z})u_{xx}(\mathbf{z}_1) + 2\varphi_\theta(\mathbf{z})u_{x\theta}(\mathbf{z}_1) + u_{\theta\theta}(\mathbf{z}_1) \\ &= \left(-u_t + u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\right)(\mathbf{z}_1). \end{aligned}$$

Hence, $u$ is a classical solution of the DCMA associated to the initial datum $u_0$. □

Lemma 6 proves that the DCMA Equation is a scalar formulation of the monodimensional heat equation (27), like two other important equations of image processing : the Mean Curvature Motion and the Affine Morphological Scale Space, which can be obtained by axiomatic formulations too (see [1]). The difference between them only comes from the intrinsic parameter of the level lines : the Euclidean abscissa for the Mean Curvature Motion, the affine abscissa for the Affine Scale space. For the DCMA, the natural parameter is the time $\theta$, which means that level lines are not considered as curves but as *graphs*. This remark will permit to prove the existence of weak solutions for the DCMA, but in certain cases only : precisely, when the level lines of the initial datum can be described by graphs.

## 3.2 Weak solutions of the DCMA

We define weak (only continuous) solutions of the DCMA as uniform limits of classical solutions.

**Definition 6** *Given a movie $u_0 \in \mathcal{C}_c^0$, we say that a map $u \in \mathcal{C}_c^{0,0}$ is a weak solution of the DCMA associated to the initial datum $u_0$ if*

$$\forall (x,\theta,t) \in \partial\Omega, \quad u(x,\theta,t) = u_0(x,\theta)$$

*and if there exists a sequence $(u^\varepsilon)_{\varepsilon>0}$ of classical solutions of the DCMA associated to the initial datum $u_0$ such that $u^\varepsilon \to u$ uniformly on $\overline{\Omega}$ when $\varepsilon \to 0$.*

**Lemma 7 (uniqueness)** *A weak solution of the DCMA associated to a given initial datum is unique.*

*Proof :*
We simply prove that the contraction property (Corollary 1) is still satisfied. Let $u$ and $\tilde{u}$ be two weak solutions of the DCMA associated to the initial data $u_0$ and $\tilde{u}_0$. Then, we can find two sequences $u^\varepsilon$ and $\tilde{u}^\varepsilon$ which converge uniformly towards $u$ and $\tilde{u}$. Writing $u_0^\varepsilon = u^\varepsilon(\cdot,\cdot,0)$ and $\tilde{u}_0^\varepsilon = \tilde{u}^\varepsilon(\cdot,\cdot,0)$, Corollary 1 ensures that $\|u^\varepsilon - \tilde{u}^\varepsilon\|_\infty \leqslant \|u_0^\varepsilon - \tilde{u}_0^\varepsilon\|$, and taking the (uniform) limits when $\varepsilon \to 0$ yields $\|u - \tilde{u}\|_\infty \leqslant \|u_0 - \tilde{u}_0\|$ as expected. □

**Proposition 2 (existence)** *Call $\overline{\mathcal{V}_c}$ the topological closure of $\mathcal{V}_c^2$ with respect to the $\|\cdot\|_\infty$ norm. Then, given $u_0 \in \overline{\mathcal{V}_c}$, there exists a unique weak solution $u$ of the DCMA associated to the initial datum $u_0$.*

*Proof :*
According to the hypothesis on $u_0$, we can find a sequence $u_0^\varepsilon \in \mathcal{V}_c^2$ which converges uniformly towards $u_0$. Then, call $u^\varepsilon$ the classical solution of the DCMA associated to the initial datum $u_0^\varepsilon$ (Proposition 1 ensures the existence of $u^\varepsilon$). Lemma 4 forces $u^\varepsilon$ to converge uniformly towards a limit $u \in \mathcal{C}_c^{0,0}$, which is by construction a weak solution of the DCMA. □

To make more precise this existence property, we now build explicit weak solutions. The construction is similar to the one used for classical solutions in the proof of Lemma 6. We first generalize Definition 4 thanks to Lemma 5.

**Definition 7** *We write $\mathcal{V}_c^0$ the space of movies $u \in \mathcal{C}_c^0$ which admit a continuous $\theta$-graph.*

**Proposition 3** *Let $u_0 \in \mathcal{V}_c^0$, and $\varphi_0$ be a $\theta$-graph of $u_0$. Define $(x,\theta,t) \mapsto \varphi(x,\theta,t)$ as the unique solution of the monodimensional heat equation (27) submitted to the boundary condition (28). Then, the unique map $u$ defined from $\varphi$ by (29) is a weak solution of the DCMA.*

*Proof :*

1. As for the definition of $u$ and its belonging to $\mathcal{C}_c^{0,0}$, the proof is already contained in Lemma 6.

2. Since $\mathcal{V}_c^0 \subset \overline{\mathcal{V}_c}$, we can consider $\tilde{u}$ the weak solution of the DCMA associated to the initial datum $u_0$, and $(u^\varepsilon)$ a sequence of classical solutions which converges uniformly towards $\tilde{u}$. Now we want to prove that $u = \tilde{u}$, or, equivalently, that

$$\forall (x,\theta,t) \in \overline{\Omega}, \quad \tilde{u}(\varphi(x,\theta,t),\theta,t) = u_0(x,0).$$

We first use a method similiar to the proof of Lemma 4 to prove that given $x_0 \in \mathbb{R}$, $\varepsilon > 0$, $\alpha > 0$ and $T > 0$, the map

$$\Lambda(\theta,t) = u^\varepsilon(\varphi(x_0,\theta,t),\theta,t) - u_0(x_0,0) - \alpha t$$

attains its maximum value over $K_T = \overline{I} \times [0,T]$, on $(\theta_0,t_0) \in K_T$ such that $\theta_0 \in \partial I$ or $t_0 = 0$, so that $\varphi(x_0,\theta_0,t_0) = \varphi_0(x_0,\theta_0)$. Writing $u_0^\varepsilon = u^\varepsilon(\cdot,\cdot,0)$, we get

$$\begin{aligned} u^\varepsilon(\varphi(x_0,\theta_0,t_0),\theta_0,t_0) &= u_0^\varepsilon(\varphi_0(x_0,\theta_0),\theta_0) \\ &\leqslant u_0(\varphi_0(x_0,\theta_0),\theta_0) + \|u_0^\varepsilon - u_0\|_\infty \\ &\leqslant u_0(x_0,0) + \|u_0^\varepsilon - u_0\|_\infty, \end{aligned}$$

$$\text{and} \qquad \forall (x,\theta,t) \in \mathbb{R} \times K_T, \qquad u^\varepsilon(\varphi(x,\theta,t),\theta,t) \leqslant u_0(x,0) + \alpha T + \|u_0^\varepsilon - u_0\|_\infty.$$

Then, sending $\alpha$ to zero and $T$ to infinity yields

$$\forall (x,\theta,t) \in \overline{\Omega}, \qquad u^\varepsilon(\varphi(x,\theta,t),\theta,t) \leqslant u_0(x,0) + \|u_0^\varepsilon - u_0\|_\infty,$$

and passing to the limit when $\varepsilon \to 0$ establishes

$$\forall (x,\theta,t) \in \overline{\Omega}, \qquad \tilde{u}(\varphi(x,\theta,t),\theta,t) \leqslant u_0(x,0).$$

A symmetrical reasoning proves that $\tilde{u}(\varphi(x,\theta,t),\theta,t) \geqslant u_0(x,0)$ as well, so that $u = \tilde{u}$ as announced. $\square$

A consequence of this characterization of weak solutions is that a weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^n$ admits a kind of velocity movie as soon as $u_0$ is locally Lipschitz in the $x$ variable. To simplify the proof, we directly assume that the whole analysis $u$ is locally Lipschitz in the $x$ variable, although it is not difficult to see that $u$ inherits this property from the initial datum $u_0$.

**Corollary 3** *Let $u$ be the weak solution of the DCMA associated to an initial datum $u_0 \in \mathcal{V}_c^0$. If $u$ is locally Lipschitz in the $x$ variable, then there exists a continuous map $v$ defined on $\Omega = \mathbb{R} \times I \times ]0,+\infty[$ such that on $\Omega$,*

$$\begin{aligned} u(x+\tau v(x,\theta,t),\theta+\tau,t) &= u(x,\theta,t) + o(\tau) \\ \text{and} \quad u(x+\tau v(x,\theta,t),\theta+\tau,t-\frac{\tau^2}{2}) &= u(x,\theta,t) + o(\tau^2). \end{aligned}$$

*Proof :*

We associate $\varphi$ to $u_0$ as in Proposition 3, and define $v$ by (25). Then,

$$\begin{aligned} u_0(x,0) &= u\left(\varphi(x,\theta+\tau,t),\theta+\tau,t\right) \\ &= u\left(\varphi(x,\theta,t)+\tau\varphi_\theta(x,\theta,t)+o(\tau),\theta+\tau,t\right) \\ &= u\left(\varphi(x,\theta,t)+\tau v(\varphi(x,\theta,t),\theta,t),\theta+\tau,t\right)+o(\tau), \end{aligned}$$

which establishes the first equality, thanks to the fact that $x \mapsto \varphi(x,\theta,t)$ is bijective. For the second one, a similar reasonning establishes that

$$u_0(x,0) = u\left(\varphi(x,\theta,t)+\tau v(\varphi(x,\theta,t),\theta,t),\theta+\tau,t-\frac{\tau^2}{2}\right)+o(\tau^2). \qquad \square$$

Notice that this property is a generalization of (22).

### 3.3 Further existence properties

In the previous sections, we did not prove the existence of (weak or classical) solutions of the DCMA in the general case, that is to say when the initial datum admits no $\theta$-graph. In fact, we do not believe that the DCMA admits a solution in general, at least a solution in the sense we defined. When the initial datum $u_0$ admits a $\theta$-graph, the DCMA is obtained by applying the linear monodimensional heat equation to the level lines of $u_0$. For an ordinary continuous map $u_0$, the level lines have no reason to be graphs in the $\theta$ variable, since to a given value of $\theta$, several values of $x$ will correspond in general. Hence, defining general solutions of the DCMA is somewhat equivalent to defining solutions of the heat equation for multi-valued data. It is in that spirit that L.C. Evans studied in [6] Equation (6) as the limit when $\varepsilon \to 0$ of the more regular equation

$$u_t = \frac{u_x^2 u_{\theta\theta} - 2u_x u_\theta u_{x\theta} + u_\theta^2 u_{xx}}{u_x^2+\varepsilon^2 u_\theta^2}. \tag{31}$$

Equation (31) admits viscosity solutions because it is more or less the Mean Curvature Motion (actually, the case $\varepsilon = 1$ is exactly the Mean Curvature Motion). He noticed that in the general case (that is, when the level lines of the initial datum are not graphs), the regularizing effects of the heat equation are so strong that the limit of approximate solutions is not continuous at scale $t=0$, because the level lines are constrained to become graphs instantaneously.

# 4 Properties of the DCMA

In this section, we investigate several properties of the DCMA. We first check the ones that are constrained by the axiomatic formulation, and then we prove that the DCMA acts as a strong smoothing process along the movement. We also establish integral estimations and associate the DCMA to a variational principle. Coming back to the original context of depth interpretation, we finally highlight geometrical properties and find a new characterization of the DCMA.

## 4.1 Checking the axioms

In order to be sure that our axiomatic formulation is consistent, we have to check that the axioms we introduced are satisfied by the DCMA. As regards the three architectural axioms (**[Semi-group]**, **[Local Comparison Principle]** and **[Regularity]**), they are direct consequences of the fact that the DCMA is given by an evolution equation of the kind $u_t = F(D^2u, Du)$, where $F$ is an elliptic operator. Now we prove that the DCMA satisfies the **[Strong Morphological Invariance]** property.

**Proposition 4** *Let $u$ be a weak solution of the DCMA and $g : \mathbb{R} \mapsto \mathbb{R}$ a continuous map. Then, $g \circ u$ is a weak solution of the DCMA.*

*Proof :*

Notice that this proposition makes sense because if $u \in \mathcal{C}^0_c$, then $g \circ u \in \mathcal{C}^0_{\tilde{c}}$ with $\tilde{c} = (g(c^-), g(c^+))$.

1. First, suppose that $u$ is a classical solution of the DCMA and that $g$ is of class $C^2$. Writing $\tilde{u} = g \circ u$, a simple computation gives

$$\begin{aligned}
\tilde{u}_{\xi\xi} &= \tilde{u}_{\theta\theta} - \frac{\tilde{u}_\theta}{\tilde{u}_x}\tilde{u}_{\theta x} + (\frac{\tilde{u}_\theta}{\tilde{u}_x})^2 \tilde{u}_{xx} \\
&= g'' \circ u \cdot (u_\theta^2 - 2u_\theta^2 + u_\theta^2) + g'(u)\left(u_{\theta\theta} - 2\frac{u_\theta}{u_x}u_{\theta x} + (\frac{u_\theta}{u_x})^2 u_{xx}\right) \\
&= g'(u) \cdot u_{\xi\xi}
\end{aligned}$$

whenever $\tilde{u}_x \neq 0$. Hence, we have $\tilde{u}_t = 0$ if $\tilde{u}_x = 0$, and $\tilde{u}_t = \tilde{u}_{\xi\xi}$ if $\tilde{u}_x \neq 0$, so that $\tilde{u}$ is a classical solution of the DCMA.

2. Now let us come back to the general case when $g$ is only continuous. Given $\varepsilon > 0$, there exists a map $g^\varepsilon \in C^2(\mathbb{R})$ such that $\|g - g^\varepsilon\|_\infty \leqslant \varepsilon$. Since the set

$$K = [-\|u\|_\infty - \varepsilon, \|u\|_\infty + \varepsilon]$$

is compact, $g$ is uniformly continuous on $K$ thanks to Heine's Theorem : in other words there exists a positive number $\alpha \leqslant \varepsilon$ such that $|g(x) - g(y)| \leqslant \varepsilon$ as soon as $|x - y| \leqslant \alpha$. Besides, we can find a classical solution $u^\varepsilon$ of the DCMA such that $\|u - u^\varepsilon\|_\infty \leqslant \alpha$. Then, we have

$$\|g \circ u - g^\varepsilon \circ u^\varepsilon\|_\infty \leqslant \|g \circ u - g \circ u^\varepsilon\|_\infty + \|g \circ u^\varepsilon - g^\varepsilon \circ u^\varepsilon\|_\infty \leqslant 2\varepsilon,$$

and $g^\varepsilon \circ u^\varepsilon$ is a classical solution of the DCMA. $\square$

As for the [**Transversal Invariance**] property, it is clearly satisfied by the DCMA since the $y$ coordinate does not even appear in its definition. Now we check the [**v-Compatibility**] property. Consider a map $h : \mathbb{R}^4 \mapsto \mathbb{R}$ such that

$$\forall u \in \mathcal{M}^1,\ R_h u \in \mathcal{M}^1 \quad \text{and} \quad v[R_h u] = v[u],$$

with $R_h u(x,y,\theta) = h\left(u(x,y,\theta),x,y,\theta\right)$. Choosing $u(x,y,\theta) = \lambda \tanh x$ (tanh meaning the hyperbolic tangent), one easily proves that $h$ is $C^1$. In addition, for any $u \in \mathcal{M}^1$ we must have $u_x h_\theta = u_\theta h_x$ in order that the condition $v[R_h u] = v[u]$ is satisfied. If we now choose $u(x,y,\theta) = \tanh x + b\theta$, we obtain $h_\theta = 0$ with $b = 0$ and then $h_x = 0$ with $b = 1$, so that we finally can write

$$h(\lambda, x, y, \theta) = f(\lambda, y).$$

Then, the relation $T_t \circ R_h = R_h \circ T_t$ is a direct consequence of Proposition 4, the $y$ coordinate being fixed. $\square$

The last two axioms, [**Galilean Invariance**] and [**Zoom Invariance**], are clearly satisfied by the DCMA thanks to Lemma 2 and Lemma 3.

## 4.2 Diffusion of the movement

In the following, $v$ is a map of class $C^2$ defined on a subset $\Omega'$ of $\Omega = \mathbb{R} \times I \times ]0,+\infty[$, and on $\Omega'$ $v$ satisfies

$$u_\theta + v u_x = 0. \tag{32}$$

This defines on $\Omega'$ the operator $\dfrac{\mathrm{D}}{\mathrm{D}\theta} := \dfrac{\partial}{\partial\theta} + v\dfrac{\partial}{\partial x}$, as well as the notation

$$f_{\xi\xi} := [D^2 f](\xi,\xi) \quad \text{with} \quad \xi = (v,1).$$

**Proposition 5** *Let $u \in \mathcal{C}_c^{n+2,1}$ be a classical solution of the DCMA, with $n \geqslant 0$. Then the first $n$ movement derivatives (velocity $v = -u_\theta/u_x$, acceleration $\Gamma = \mathrm{D}v/\mathrm{D}\theta$, $\ldots \mathrm{D}^{n-1}v/\mathrm{D}\theta^{n-1}$) are diffused in the same direction as $u$, that is*

$$\forall k \in \{0, \ldots n-1\}, \qquad \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_t = \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_{\xi\xi} \quad \text{whenever} \quad u_x \neq 0.$$

*In particular, the apparent velocity $v$ satisfies*

$$v_t = v_{\theta\theta} + 2v v_{\theta x} + v^2 v_{xx} \quad \text{whenever} \quad u_x \neq 0.$$

To establish this property, it is interesting to introduce the formalism of the Lie brackets associated to the partial derivatives $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial t}$, which commute together, and to the total derivative $\frac{\mathrm{D}}{\mathrm{D}\theta} = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}$. We compute

$$[\frac{\partial}{\partial x}, \frac{\mathrm{D}}{\mathrm{D}\theta}] = \frac{\partial}{\partial x}\frac{\mathrm{D}}{\mathrm{D}\theta} - \frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}(\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x}) - (\frac{\partial}{\partial\theta} + v\frac{\partial}{\partial x})\frac{\partial}{\partial x} = v_x\frac{\partial}{\partial x}.$$

One easily checks as well that

$$[\frac{\partial}{\partial\theta},\frac{\mathrm{D}}{\mathrm{D}\theta}]=v_\theta\frac{\partial}{\partial x}\quad\text{and}\quad[\frac{\partial}{\partial t},\frac{\mathrm{D}}{\mathrm{D}\theta}]=v_t\frac{\partial}{\partial x}.$$

This way, we can expand the $f_{\xi\xi}=[D^2f](\xi,\xi)$ notation into

$$\begin{aligned}
(\ )_{\xi\xi} &= \frac{\partial^2}{\partial\theta^2}+2v\frac{\partial^2}{\partial\theta\partial x}+v^2\frac{\partial^2}{\partial x^2}\\
&= (\frac{\partial}{\partial\theta}+v\frac{\partial}{\partial x})\frac{\partial}{\partial\theta}+v(\frac{\partial}{\partial\theta}+v\frac{\partial}{\partial x})\frac{\partial}{\partial x}\\
&= \frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial\theta}+v\frac{\mathrm{D}}{\mathrm{D}\theta}\frac{\partial}{\partial x}\\
&= \frac{\mathrm{D}}{\mathrm{D}\theta}(\frac{\partial}{\partial\theta}+v\frac{\partial}{\partial x})-\frac{\mathrm{D}v}{\mathrm{D}\theta}\frac{\partial}{\partial x}
\end{aligned}$$

and finally we get, writing $\Gamma=\frac{\mathrm{D}v}{\mathrm{D}\theta}$, the equality $(\ )_{\xi\xi}=\frac{\mathrm{D}^2}{\mathrm{D}\theta^2}-\Gamma\frac{\partial}{\partial x}$. In particular, if we write $\psi=\frac{\mathrm{D}\Gamma}{\mathrm{D}\theta}$ the total derivative of $\Gamma$, we have $v_{\xi\xi}=\psi-\Gamma v_x$.

**Lemma 8** *Independently of any evolution equation, on $\Omega'$ we have*

$$[\frac{\partial}{\partial t}-(\ )_{\xi\xi},\frac{\mathrm{D}}{\mathrm{D}\theta}]=(v_t-v_{\xi\xi})\frac{\partial}{\partial x}. \tag{33}$$

*Proof :*

We compute the Lie bracket

$$\begin{aligned}
[(\,)_{\xi\xi},\frac{\mathrm{D}}{\mathrm{D}\theta}] &= [\frac{\mathrm{D}^2}{\mathrm{D}\theta^2}-\Gamma\frac{\partial}{\partial x},\frac{\mathrm{D}}{\mathrm{D}\theta}]=[\frac{\mathrm{D}^2}{\mathrm{D}\theta^2},\frac{\mathrm{D}}{\mathrm{D}\theta}]-[\Gamma\frac{\partial}{\partial x},\frac{\mathrm{D}}{\mathrm{D}\theta}]\\
&= 0+\frac{\mathrm{D}\Gamma}{\mathrm{D}\theta}\frac{\partial}{\partial x}-\Gamma[\frac{\partial}{\partial x},\frac{\mathrm{D}}{\mathrm{D}\theta}]=(\psi-\Gamma v_x)\frac{\partial}{\partial x}\\
&= v_{\xi\xi}\frac{\partial}{\partial x}.
\end{aligned}$$

Now, by linearity, we get as announced

$$[\frac{\partial}{\partial t}-(\,)_{\xi\xi},\frac{\mathrm{D}}{\mathrm{D}\theta}]=[\frac{\partial}{\partial t},\frac{\mathrm{D}}{\mathrm{D}\theta}]-[(\,)_{\xi\xi},\frac{\mathrm{D}}{\mathrm{D}\theta}]=(v_t-v_{\xi\xi})\frac{\partial}{\partial x}.\qquad\square$$

*Proof of Proposition 5 :*

We take $\Omega'=\{\mathbf{z}\in\Omega,\ u_x(\mathbf{z})\neq 0\}$, so that $v$ is uniquely defined by (32) on $\Omega'$. Applying (33) to $u$ yields

$$\left(\frac{\partial}{\partial t}-(\ )_{\xi\xi}\right)\frac{\mathrm{D}u}{\mathrm{D}\theta}+\frac{\mathrm{D}}{\mathrm{D}\theta}(u_t-u_{\xi\xi})=(v_t-v_{\xi\xi})u_x. \tag{34}$$

As $u$ satisfies $\frac{\mathrm{D}u}{\mathrm{D}\theta}=0$ as well as $u_t=u_{\xi\xi}$ on $\Omega'$ ($u$ is solution of the DCMA), the left term of (34) is zero. Hence, on $\Omega'$ we have $v_t=v_{\xi\xi}$ as announced in Proposition 5. This proves that the right term of (33) is zero on $\Omega'$, so that

$$[\frac{\partial}{\partial t}-(\ )_{\xi\xi},\frac{\mathrm{D}}{\mathrm{D}\theta}]=0\quad\text{whenever}\quad u_x\neq 0.$$

Consequently, for any $q : \Omega' \to \mathbb{R}$ of class $C^3$ satisfying $q_t = q_{\xi\xi}$, we have

$$\left(\frac{\mathrm{D}q}{\mathrm{D}\theta}\right)_t = \left(\frac{\mathrm{D}q}{\mathrm{D}\theta}\right)_{\xi\xi} \quad \text{whenever} \quad u_x \neq 0.$$

Thus, a simple induction proves that the diffusion equation $q_t = q_{\xi\xi}$ is satisfied by all successive total derivatives of $v$ of class $C^2$, that is, $\frac{\mathrm{D}v}{\mathrm{D}\theta}, \ldots, \frac{\mathrm{D}^n v}{\mathrm{D}\theta^n}$. □

Now we would like to generalize Proposition 5 to the whole $\Omega$, i.e. even at points where $u_x$ vanishes.

**Proposition 6** *If* $u \in \mathcal{V}_c^{n+3,1}$, *then there exists a velocity map* $v$ *associated to* $u$ *which satisfies,* **on the whole** $\Omega$,

$$\forall k \in \{0, \ldots n\}, \qquad \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_t = \left(\frac{\mathrm{D}^k v}{\mathrm{D}\theta^k}\right)_{\xi\xi}. \tag{35}$$

*Moreover, if* $I = ]\theta_1, \theta_2[$, *then*

$$\forall (x, i, t) \in \mathbb{R} \times \{1, 2\} \times ]0, +\infty[ \qquad \Gamma(x, \theta_i, t) = 0. \tag{36}$$

*Proof :*

Define $\varphi$ as in Lemma 6, and consider the velocity map $v$ defined by

$$v(\varphi(x, \theta, t), \theta, t) = \varphi_\theta(x, \theta, t). \tag{37}$$

1. We get, writing $\mathbf{z}_0 = (x, \theta, t)$ and $\mathbf{z}_1 = (\varphi(x, \theta, t), \theta, t)$,

$$v_t(\mathbf{z}_1) = \varphi_{\theta t}(\mathbf{z}_0) - \varphi_t(\mathbf{z}_0) v_x(\mathbf{z}_1) = \varphi_{\theta\theta\theta}(\mathbf{z}_0) - \varphi_{\theta\theta}(\mathbf{z}_0) v_x(\mathbf{z}_1),$$

$$\text{while} \quad \varphi_{\theta\theta}(\mathbf{z}_0) = v_\theta(\mathbf{z}_1) + \varphi_\theta(\mathbf{z}_0) v_x(\mathbf{z}_1)$$

$$\text{and} \quad \varphi_{\theta\theta\theta}(\mathbf{z}_0) = v_{\theta\theta}(\mathbf{z}_1) + 2\varphi_\theta(\mathbf{z}_0) v_{\theta x}(\mathbf{z}_1) + \varphi_\theta^2(\mathbf{z}_0) v_{xx}(\mathbf{z}_1) + \varphi_{\theta\theta}(\mathbf{z}_0) v_x(\mathbf{z}_1).$$

$$\begin{aligned} \text{Hence,} \quad v_t(\mathbf{z}_1) &= v_{\theta\theta}(\mathbf{z}_1) + 2\varphi_\theta(\mathbf{z}_0) v_{\theta x}(\mathbf{z}_1) + \varphi_\theta^2(\mathbf{z}_0) v_{xx}(\mathbf{z}_1) \\ &= \left(v_{\theta\theta} + 2 v v_{\theta x} + v^2 v_{xx}\right)(\mathbf{z}_1) = v_{\xi\xi}(\mathbf{z}_1) \end{aligned}$$

as expected. This proves that the right term of (33) is identically zero on the whole $\Omega$, so that this diffusion property extends to the successive total derivatives of $v$ as we noticed in the proof of Proposition 5.

2. Differentiating (37) with respect with $\theta$, we get $\Gamma(\varphi(x, \theta, t), \theta, t) = \varphi_{\theta\theta}(x, \theta, t)$, so that for any $(x, i, t)$ in $\mathbb{R} \times \{1, 2\} \times ]0, +\infty[$ we have

$$\Gamma(\varphi(x, \theta_i, t), \theta_i, t) = \varphi_t(x, \theta_i, t) = \frac{\partial}{\partial t}\varphi(x, \theta_i, t) = \frac{\partial}{\partial t}\varphi(x, \theta_i, 0) = 0. \qquad \square$$

**Remark :** If $u \in \mathcal{C}_c^{0,0}$ is a weak solution of the DCMA, locally Lipschitz in the $x$ variable, it is possible to establish an equivalent result in the continuation of Corollary 3, provided that we substitute the total derivative $\frac{\mathrm{D}}{\mathrm{D}\theta}$ by the Lie derivative

$$f_\xi(x,\theta,t) := \left(\frac{d}{d\tau} f(x+\tau v(x,\theta,t), \theta+\tau, t)\right)_{\tau=0}.$$

From Corollary 3 we know that there exists a velocity map $v$ (i.e. such that $u_\xi = 0$), defined on $\Omega$, which also constrains

$$u(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = u(x,\theta,t) + o(\tau^2).$$

Then, it is not difficult to show that

$$v(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = v(x,\theta,t) + \tau v_\xi(x,\theta,t) + o(\tau^2).$$

More generally, the successive Lie derivatives of $v$ along the movement are well defined ($\Gamma = v_\xi$, $\psi = \Gamma_\xi$, ..., $v^{[n+1]} = (v^{[n]})_\xi$, ...) and satisfy

$$v^{[n]}(x+\tau v(x,\theta,t), \theta+\tau, t-\frac{\tau^2}{2}) = v^{[n]}(x,\theta,t) + \tau v^{[n+1]}(x,\theta,t) + o(\tau^2).$$

## 4.3 A conservation law

We would like to consider integrals like

$$\int_I \int_{-\infty}^{+\infty} u(x,\theta)\, dx d\theta.$$

To simplify the results, we are going to work on compactly supported movies (that is, maps $u$ which are null outside a compact set of $\mathbb{R} \times \overline{I}$) which is not very restrictive physically speaking.

**Lemma 9** *A compactly supported movie $u \in \mathcal{V}_0^n$ ($n \geqslant 1$) admits a compactly supported velocity map.*

*Proof :*
Suppose that $u(x,\theta) = 0$ when $|x| \geqslant R$ and let $v$ be a velocity map of $u$. There exists a map $\phi \in C^\infty(\mathbb{R})$ such that $\phi(x) = 0$ if $|x| \geqslant R+1$ and $\phi(x) = 1$ if $|x| \leqslant R$. Thus, the map $\tilde{v} : (x,\theta) \mapsto \phi(x) \cdot v(x,\theta)$ is a velocity map of $u$ because $u_\theta = u_x = 0$ when $|x| > R$. Last, it is clear that $\tilde{v}$, as well as $v$, is bounded and of class $C^{n-1}$. □

**Proposition 7** *Let $u$ be the (weak or classical) solution of the DCMA associated to a compactly supported initial datum $u_0 \in \mathcal{V}_0^n$. Then,*

$$\exists R > 0,\ \forall (x,\theta,t) \in \mathbb{R} \times \overline{I} \times [0,+\infty[, \qquad |x| \geqslant R+t \ \Rightarrow\ u(x,\theta,t) = 0. \tag{38}$$

*and if $n \geqslant 1$, $u$ admits a velocity map which satisfies the same conclusion.*

*Proof :*

This is a simple consequence of (30). Recall that the solution $u$ of the DCMA can be defined by

$$\forall (x,\theta,t)\in\overline{\Omega},\ \ u(\varphi(x,\theta,t),\theta,t)=u_0(x,0),$$

where $\varphi$ satisfies

$$\exists C,\ \forall (x,\theta,t)\in\overline{\Omega},\quad |\varphi(x,\theta,t)|\geqslant |x|-C-t$$

thanks to (30). But since $u_0$ is compactly supported, there exists $R>0$ such that $u_0(x,\theta)=0$ as soon as $|x|\geqslant R-C$. Then, $u(x,\theta,t)=0$ as soon as $|x|\geqslant t+R$. □

**Proposition 8 (Light Energy Conservation)** *Let $u\in\mathcal{V}_0^{2,1}$ be the classical solution of the DCMA associated to a compactly supported initial datum. Suppose that*

*(a) either $I=S^1$,*

*(b) or $I\neq S^1$ and $\forall (x,i)\in\mathbb{R}\times\{1,2\},\quad u(x,\theta_i,0)=0.$*

*Then, the light energy at scale $t$, defined by*

$$I(t)=\frac{1}{2}\iint u^2(x,\theta,t)\,dxd\theta,$$

*is independent of $t$.*

*Proof :*

We take the convention $(\theta_1,\theta_2)=(0,2\pi)$ if $I=S^1$, and remark that if $I=]\theta_1,\theta_2[$, then the boundary condition on $u$ implies

$$\forall (x,i,t)\in\mathbb{R}\times\{1,2\}\times[0,+\infty[\quad u(x,\theta_i,t)=u(x,\theta_i,0)=0$$

thanks to Condition (b), so that

$$\forall (x,i,t)\in\mathbb{R}\times\{1,2\}\times[0,+\infty[\quad u_x(x,\theta_i,t)=\frac{\partial}{\partial x}u(x,\theta_i,t)=0.$$

In the following, $v$ is a velocity map associated to $u$. Since $u(\cdot,\cdot,t)$ is compactly supported thanks to Proposition 7, the integral defining $I(t)$ is taken on a compact set. Consequently, as $u\in\mathcal{C}_0^{2,1}$, $I$ is derivable and we can derive under the integral symbol to obtain

$$I'(t)=\iint uu_t\,dxd\theta=-\iint uu_x(v_\theta+vv_x)\,dxd\theta=-\iint uu_xv_\theta-uu_\theta v_x\,dxd\theta.$$

By integrating by parts, we get

$$I'(t)=-\int[uu_xv]_{\theta_1}^{\theta_2}\,dx+\int[uu_\theta v]_{-\infty}^{+\infty}\,d\theta+\iint(uu_x)_\theta v-(uu_\theta)_xv\,dxd\theta.$$

The first term is zero thanks to (a) or (b), the second one is zero because $u(\cdot,\cdot,t)$ is compactly supported and $v$ is bounded, and the third one is evidently zero. Hence, $I(t)$ does not depend on $t$. □

## 4.4 A variational principle

**Proposition 9** *Let $u \in \mathcal{V}_0^{4,1}$ be the classical solution of the DCMA associated to a compactly supported initial datum. Then,*

$$E(t) := \frac{1}{2} \iint \Gamma^2(x,\theta,t)\, dx d\theta$$

*decreases with scale and we have*

$$\frac{dE}{dt}(t) = -\iint \left(\frac{\mathrm{D}\Gamma}{\mathrm{D}\theta}\right)^2 dx d\theta. \tag{39}$$

*Proof :*

In all the following, $v$ is a velocity field of $u$ satisfying (38). Notice that

$$\Gamma_{\xi\xi} = \frac{\mathrm{D}^2\Gamma}{\mathrm{D}\theta^2} - \Gamma\Gamma_x = \frac{\mathrm{D}\Psi}{\mathrm{D}\theta} - \Gamma\Gamma_x \quad \text{with} \quad \Psi = \frac{\mathrm{D}\Gamma}{\mathrm{D}\theta} = \Gamma_\theta + v\Gamma_x.$$

We compute the derivative of $E(t)$,

$$E'(t) = \iint \Gamma\Gamma_{\xi\xi}\, dx d\theta = \iint \Gamma(\Psi_\theta + v\Psi_x - \Gamma\Gamma_x)\, dx d\theta = \iint \Gamma\Psi_\theta + (v\Gamma)\Psi_x - \Gamma^2\Gamma_x\, dx d\theta.$$

Integrating by parts the first two terms yields

$$E'(t) = \int [\Gamma\psi]_{\theta_1}^{\theta_2} dx + \int [v\Gamma\psi]_{-\infty}^{+\infty} d\theta - \iint \Gamma_\theta\Psi + (v\Gamma)_x\Psi + \Gamma^2\Gamma_x\, dx d\theta.$$

The first bracket is zero thanks to (36) (or thanks to the periodicity of $\Gamma\Psi$ if $I = S^1$), and the second one is zero because $v\Gamma\Psi$ is compactly supported at any scale $t$. Hence,

$$\begin{aligned} E'(t) &= -\iint \Gamma_\theta\Psi + (v\Gamma)_x\Psi + \Gamma^2\Gamma_x\, dx d\theta \\ &= -\iint \Psi^2 dx d\theta - \iint v_x\Gamma\Psi\, dx d\theta + \iint \Gamma^2\Gamma_x\, dx d\theta. \end{aligned}$$

But as

$$\iint \Gamma^2\Gamma_x\, dx d\theta = \frac{1}{3}\iint \frac{\partial}{\partial x}(\Gamma^3)\, dx d\theta = 0$$

(because $\Gamma$ is compactly supported at any scale $t$), the second term of (40) is

$$\begin{aligned} B(t) := \iint v_x\Gamma\Psi\, dx d\theta &= \iint v_x\Gamma\Psi - \Gamma^2\Gamma_x\, dx d\theta \\ &= \iint \Gamma(\Gamma_\theta v_x + v v_x\Gamma_x - \Gamma_x v_\theta - v v_x\Gamma_x)\, dx d\theta \\ &= \frac{1}{2}\iint (2\Gamma\Gamma_\theta)v_x - (2\Gamma\Gamma_x)v_\theta\, dx d\theta. \end{aligned}$$

Then, another integration by parts yields

$$2B(t) = \int \left[\Gamma^2 v_x\right]_{\theta_1}^{\theta_2} dx + \int \left[\Gamma^2 v_\theta\right]_{-\infty}^{+\infty} d\theta - \iint \Gamma^2(v_{x\theta} - v_{\theta x})\, dx d\theta = 0.$$

Finally, coming back to (40), we obtain, as announced,

$$E'(t) = -\iint \Psi^2 dx d\theta \leqslant 0. \qquad \square$$

At this point, it is natural to wonder whether (39) results from a variational principle. Let us consider the functional

$$\mathcal{E}(v) = \frac{1}{2} \iint (v_\theta + v v_x)^2 \, dx d\theta,$$

defined on compactly supported movies of class $C^2$. Differentiating $\mathcal{E}$ yields

$$D_v\mathcal{E}(h) = \iint (v_\theta + v v_x)(h_\theta + (vh)_x) \, dx d\theta = \iint \Gamma h_\theta + (\Gamma v) h_x + \Gamma v_x h \, dx d\theta.$$

By integrating by parts the first two terms, we get, assuming that (36) is satisfied by $\Gamma$ if $I \neq S^1$,

$$D_v\mathcal{E}(h) = \iint -\Gamma_\theta h - \Gamma_x v h \, dx d\theta = -\iint \frac{\mathrm{D}\Gamma}{\mathrm{D}\theta} h \, dx d\theta = -\iint \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} \cdot h \, dx d\theta.$$

Hence, the canonical evolution equation associated to the minimization of $\mathcal{E}$ would be

$$\frac{\partial v}{\partial t} = \frac{\mathrm{D}^2 v}{\mathrm{D}\theta^2} = v_{\xi\xi} + \Gamma v_x.$$

Because of the last term $\Gamma v_x$, we can see that the equation $v_t = v_{\xi\xi}$ induced by the DCMA is not exactly the evolution equation associated to the minimization of $\mathcal{E}$. However, Proposition 9 showed that for the DCMA evolution,

$$D_v\mathcal{E}(\frac{\partial v}{\partial t}) = \frac{d}{dt} E(t) = -\iint \left(\frac{\mathrm{D}^2 v}{\mathrm{D}\theta}\right)^2 \, dx d\theta$$

as if it was the case.

## 4.5 Interpretation for the observed scene

In this section, we do not omit the $y$ variable any longer. We recall that $(S^1)^\star = \mathbb{R}$ and $]\theta_1, \theta_2[^\star = ]\theta_1, \theta_2[$.

### 4.5.1 Ideal movies

**Definition 8** *A movie* $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$ *is* **ideal** *if one can find three maps* $(C, Z, U) \in C^0(I^\star) \times C^0(\mathbb{R}^2) \times C^0(\mathbb{R}^2)$ *such that*

$$\begin{array}{rccl} \Pi & : & \mathbb{R}^2 \times \overline{I^\star} & \to \mathbb{R}^2 \times \overline{I} \\ & & (X, Y, \theta) & \mapsto \left(\dfrac{X - C(\theta)}{Z(X,Y)}, \dfrac{Y}{Z(X,Y)}, \theta\right) \end{array}$$

*is bijective and*

$$\forall (X, Y, \theta) \in \mathbb{R}^2 \times \overline{I^\star}, \qquad u \circ \Pi(X, Y, \theta) = U(X, Y). \tag{40}$$

In other terms, a movie is ideal if it can be interpreted as the perfect observation of a scene $Z(X,Y), U(X,Y)$ (depth and Lambertian luminance) by a unit focal length camera submitted to the movement $X = C(\theta)$. In this definition, occlusions are forbidden because $\Pi$ is constrained to be bijective. If $I = S^1$, the natural injection $\mathbb{R} \hookrightarrow S^1$ is implicit in the definition.

It is important to notice that the interpretation of a movie is never unique. Indeed, if $(C, Z, U)$ is an interpretation of $u$, then $(\lambda C, \lambda Z \circ D_\lambda, U \circ D_\lambda)$ with $D_\lambda : (X,Y) \mapsto (X/\lambda, Y/\lambda)$ is another interpretation of $u$. This ambiguity is well-known as the aperture problem : if one do not know the focal length of a camera, the depth on the movie it produces can at most be recovered up to a multiplicative factor . Moreover, it is clear that the depth cannot be recovered in regions $(X, Y, Z(X,Y))$ where $U$ is constant.

### 4.5.2 Differential characterization of ideal movies

**Proposition 10** *If a movie is ideal and allows a derivable movement interpretation, then it admits a velocity map $v$, and in any point where $v$ is $C^2$ we have*

$$v \cdot \nabla\Gamma - \Gamma \cdot \nabla v = \mathbf{0}, \tag{41}$$

*where $\nabla = (\partial/\partial x, \partial/\partial y)$ is the spatial gradient operator.*

*Proof :*

Let $(C, Z, U)$ be an interpretation of $u$ such that $C$ is derivable. We define a unique movie $v : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$ by

$$v \circ \Pi(X,Y,\theta) = \frac{-C'(\theta)}{Z(X,Y)}. \tag{42}$$

Then, differentiating (40) with respect to $\theta$ yields $(vu_x + u_\theta) \circ \Pi = 0$, so that $v$ is a velocity map of $u$ as announced. Now, anywhere $v$ is $C^2$ we have

$$(vv_x + v_\theta) \circ \Pi(X,Y,\theta) = \frac{-C''(\theta)}{Z(X,Y)},$$

which can be combined with (42) to yield

$$C'(\theta) \cdot \Gamma \circ \Pi(X,Y,\theta) = C''(\theta) \cdot v \circ \Pi(X,Y,\theta)$$

because $Z$ does not vanish. But if $C'(\theta) \neq 0$, then $v \neq 0$ and $\Gamma/v$ does not depend on $x$, so that, as announced,

$$\mathbf{0} = \nabla \frac{\Gamma}{v} = \frac{v\nabla\Gamma - \Gamma\nabla v}{v^2}.$$

If $C'(\theta) = 0$ and $C''(\theta) \neq 0$, the same reasoning holds for the map $v/\Gamma$. Last, if $C'(\theta) = 0$ and $C''(\theta) = 0$, then (41) is clearly satisfied because $v = \Gamma = 0$. $\square$

Now a natural question arises : does an ideal movie remain ideal when it evolves according to the DCMA ? To prove that the answer is yes, we could show that the differential invariant of (41) remains null if it is null at initial scale. In fact, we state a better property by interpreting exactly the evolution of an ideal movie.

### 4.5.3 Evolution of ideal movies

**Theorem 3** *Let $u_0 \in \mathcal{C}_c^2$ be an ideal movie associated with an interpretation $(Z_0(\cdot), U_0(\cdot), C_0(\cdot))$ such that*

$$\exists A, B,\ \forall \theta \in I^\star, \quad |C_0(\theta)| \leqslant A + B|\theta|.$$

*Then the classical solution $u$ of the DCMA defined from the initial datum $u_0$ is a multiscale collection of ideal movies $((u(\cdot,t))_{t\geqslant 0}$. Moreover, these movies can be interpreted as $(Z_0(\cdot), U_0(\cdot), C(\cdot,t))$, where $C(\cdot,\cdot)$ is defined by*

$$\begin{cases} C_t = C_{\theta\theta} & \text{on} \quad \Omega = I^\star \times ]0,+\infty[, \\ C(\theta,t) = C_0(\theta) & \text{on} \quad \partial\Omega. \end{cases}$$

*Proof :*

1. Let $C$ be the solution of the heat equation as specified in the theorem. The map

$$\begin{array}{rccl} \Pi & : & \mathbb{R}^2 \times \overline{I^\star} & \rightarrow \mathbb{R}^2 \times \overline{I} \times [0,+\infty[ \\ & & (X,Y,\theta,t) & \mapsto \left( \dfrac{X - C(\theta,t)}{Z_0(X,Y)}, \dfrac{Y}{Z_0(X,Y)}, \theta, t \right) \end{array}$$

is bijective because the heat equation satisfies the comparison principle. Hence, we can define a collection of ideal movies $\tilde{u}(\cdot,t)$ from

$$\tilde{u} \circ \Pi(X,Y,\theta) = U_0(X,Y), \tag{43}$$

2. First we check that $\tilde{u}$ is $C^2$. Choose $(X_0, Y_0, \theta_0, t_0) \in \mathbb{R}^2 \times \overline{I^\star} \times ]0,+\infty[$, and write $(X(h), Y(h))$ the unique element of $\mathbb{R}^2$ such that

$$\Pi(X(h), Y(h), \theta_0, t_0) = \Pi(X_0, Y_0, \theta_0, t_0) + (h,0,0) = (x_0 + h, y_0, \theta_0, t_0).$$

We have, for any $\theta$ and $h$,

$$\tilde{u}(x_0 + h, y_0, \theta_0, t_0) = U_0(X(h), Y(h)) = u_0(x_0 + h + \frac{C(\theta_0,t_0) - C_0(\theta)}{Z_0(X(h),Y(h))}, y_0, \theta).$$

Now, there exists a unique $\theta_1$ such that $C_0(\theta_1) = C(\theta_0, t_0)$, so that we finally get

$$\tilde{u}(x_0 + h, y_0, \theta_0, t_0) = u_0(x_0 + h, y_0, \theta_1).$$

This proves that $\tilde{u}$ is, like $u_0$, derivable with respect to $x$. A similar reasoning establishes that $u \in \mathcal{C}_c^{2,1}$.

3. Now we prove that $u = \tilde{u}$. If we compute the derivatives of (43) with respect to $\theta$ and $t$, we obtain

$$-\frac{C'(\theta,t)}{Z_0(X,Y)} \tilde{u}_x \circ \Pi + \tilde{u}_\theta \circ \Pi = 0 \quad \text{and} \quad -\frac{C''(\theta,t)}{Z_0(X,Y)} \tilde{u}_x \circ \Pi + \tilde{u}_t \circ \Pi = 0.$$

If $\tilde{u}_x \circ \Pi = 0$, then $\tilde{u}_t \circ \Pi = 0$, and if $\tilde{u}_x \circ \Pi \neq 0$, eliminating $C$ yields

$$\tilde{u}_t \circ \Pi = \frac{\tilde{u}_x \circ \Pi}{Z_0(X,Y)} \frac{\partial}{\partial\theta} \left( Z_0(X,Y) \frac{\tilde{u}_\theta}{\tilde{u}_x} \circ \Pi \right) = \left[ \tilde{u}_x \frac{\mathrm{D}}{\mathrm{D}\theta} (\frac{\tilde{u}_\theta}{\tilde{u}_x}) \right] \circ \Pi = \tilde{u}_{\xi\xi} \circ \Pi.$$

Hence, $\tilde{u}$ is a classical solution of the DCMA submitted to the same boundary constraint as $u$. Since these conditions define a unique solution, we can deduce that $u = \tilde{u}$, which proves that each movie $u(\cdot,\cdot,\cdot,t)$ is ideal and that we can choose the interpretation announced in the theorem. □

### 4.5.4 Characterization of the DCMA

We now give another justification for the DCMA equation obtained in Theorem 2.

**Theorem 4** *The DCMA is, up to a rescaling, the only multiscale analysis satisfying the architectural axioms, the* **[v-compatibility]** *axiom, and such that an ideal movie* $(C_0, Z, U)$ *is transformed into a sequence of ideal movies* $(C(t), Z, U)$ *such that* $C(t)$ *depends linearly on* $C_0$.

*Sketch of the proof (see [14] for a complete proof) :* The first step consists in establishing the relationship between the derivatives of a map $F$ in the scene referential $(X, Y, \Theta, T)$ and its derivatives in the image referential $(x, y, \theta, t)$. Applied to the depth $Z$, this permits to show that

$$Z_T = \frac{-Z_x C_t + Z Z_t}{Z + x Z_x + y Z_y} = Z_X \left( \frac{Z Z_t}{Z_x} - C_t \right).$$

The second step is to write the evolution (7) of the ideal movie $u$ as

$$\frac{\partial u}{\partial t} = u_\theta F\left(\frac{V_\theta}{V}, -\frac{V}{Z}\right),$$

and to obtain an equation between $Z_x$, the partial derivatives of $F$ and the successive derivatives of $C$ knowing that $Z_T = 0$. Then, the formal independancy of $Z_x$ and the hypothesis on the linearity of the evolution of $C$ allows to conclude. □

# 5 Numerical scheme

In order to apply the DCMA evolution to real movies, we need to devise a numerical scheme. A "naive" discretization of the partial derivatives of $u$ cannot be used, because in practice it is well known that the time discretization is not thin enough. Moreover, such a discretization is not likely to satisfy the axioms that we imposed to the DCMA. This is the reason why we focus our attention on an inf-sup scheme. To this end, given a movie $u : \mathbb{R}^2 \times \overline{I} \to \mathbb{R}$, we define

$$\begin{aligned} IS_h u(x_0, y_0, \theta_0) &= \inf_{v \in \mathbb{R}} \sup_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ SI_h u(x_0, y_0, \theta_0) &= \sup_{v \in \mathbb{R}} \inf_{-h \leqslant \theta \leqslant h} u(x_0 + v\theta, y_0, \theta_0 + \theta), \\ \text{and} \quad T_h u &= \frac{1}{2} \left( IS_h u + SI_h u \right). \end{aligned}$$

We have a consistency result (see [14] for a proof) at points where $u_x$ does not vanish.

**Theorem 5** *If* $u$ *is a bounded movie locally* $C^3$ *near* $\mathbf{z}_0$*, with* $u_x(\mathbf{z}_0) \neq 0$*, then*

$$T_h u(\mathbf{z}_0) = u(\mathbf{z}_0) + \frac{1}{2} h^2 u_{\xi\xi}(\mathbf{z}_0) + O(h^3),$$

*and the* $O(h^3)$ *is uniform in a neighborhood of* $\mathbf{z}_0$.

Theorem 5 proves the consistency of the numerical scheme given by the iteration of $T_h$ with respect to the DCMA evolution. Due to the $h^2$ coefficient in the expansion of $T_h$, it is natural to consider the numerical scheme which associates, to a given movie $u_0$ and a scale $t \geqslant 0$, the sequence of movies $(u_{n,t})_{n\geqslant 1}$ given by

$$u_n = (T_{h_n})^n \, u_0, \quad \text{with} \quad h_n = \sqrt{2t/n},$$

and satisfying the boundary constraint $u_n(x,y,\theta) = u_0(x,y,\theta)$ on $\partial(\mathbb{R}^2 \times I)$. As for the convergence, we could hope to prove that $u_n$ converges towards the DCMA of $u_0$ when the partial derivative of $u_0$ with respect to $x$ never vanishes (but this would not be very useful). In the general case, the existence of a solution is not ensured, even if in practice the discrete nature of computer data ensures the convergence of the algorithm. In fact, at singular points where no velocity can be defined, the numerical scheme blows up, as suggested by the following

**Proposition 11** *Let $P$ be a polynomial with degree at most two such that $P_x(x_0,\theta_0) = 0$. Then, in $(x_0,\theta_0)$ we have, as $h \to 0$,*

$$T_h P = P + \frac{h}{2}\,|P_\theta|\,\mathrm{sgn}(P_{xx}) + O(h^2)$$

Proposition 11 suggests that the numerical scheme we proposed may induce a projection of the initial datum from $\mathcal{C}_c^0$ to $\mathcal{V}_c^0$, defined by the asymptotic state of

$$u_t = \begin{cases} |u_\theta|\mathrm{sgn}(u_{xx}) & \text{if} \quad u_x = 0, \\ 0 & \text{else.} \end{cases}$$

Notice that if we follow Evans (see [6]) and consider the DCMA as the limit of (31), we obtain a different projection operator in general.

It is important to notice the extreme simplicity of the algorithm we presented : in particular, it can be implemented very easily on a massive parallel machine. Our optimized code in C language for one iteration consists of only 23 instructions. This algorithm is currently being tested to perform real-time depth recovery in case of aerial observations of fixed landscapes.

# 6 Conclusion

In this paper, we focused our attention on a simple — but quite representative — case of the “structure from motion” problem, assuming that the camera motion was straight and parallel to the focal plane. We showed that the depth recovery problem is closely related to a particular degenerate parabolic PDE, that we called Depth Compatible Multiscale Analysis (DCMA) for it preserves the depth interpretation of “ideal” movies — that is to say, exact observations of fixed scenes. Like other similar diffusion equations (e.g. Mean Curvature Motion), the DCMA can be viewed as a geometric formulation of the heat

Equation on the level lines of a scalar map. It is not directly handled by the classical theory of viscosity solutions, because it presents a strong singularity inherent to the irreversibility of the time variable. However, it is possible to define weak solutions in order to ensure uniqueness and existence results under reasonable assumptions. Several properties of solutions can also be interpreted from the "structure from motion" point of view; for example, the convergence of solutions to "ideal movies" can be theoretically proven and was checked on experiments, using a simple numerical scheme. The existence of the DCMA and this associated scheme seems to be an encouraging clue that a mathematical and a numerical solution of the "structure from motion" problem should exist, even if the problem of handling occlusions still remains.

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# Appendix : experiments

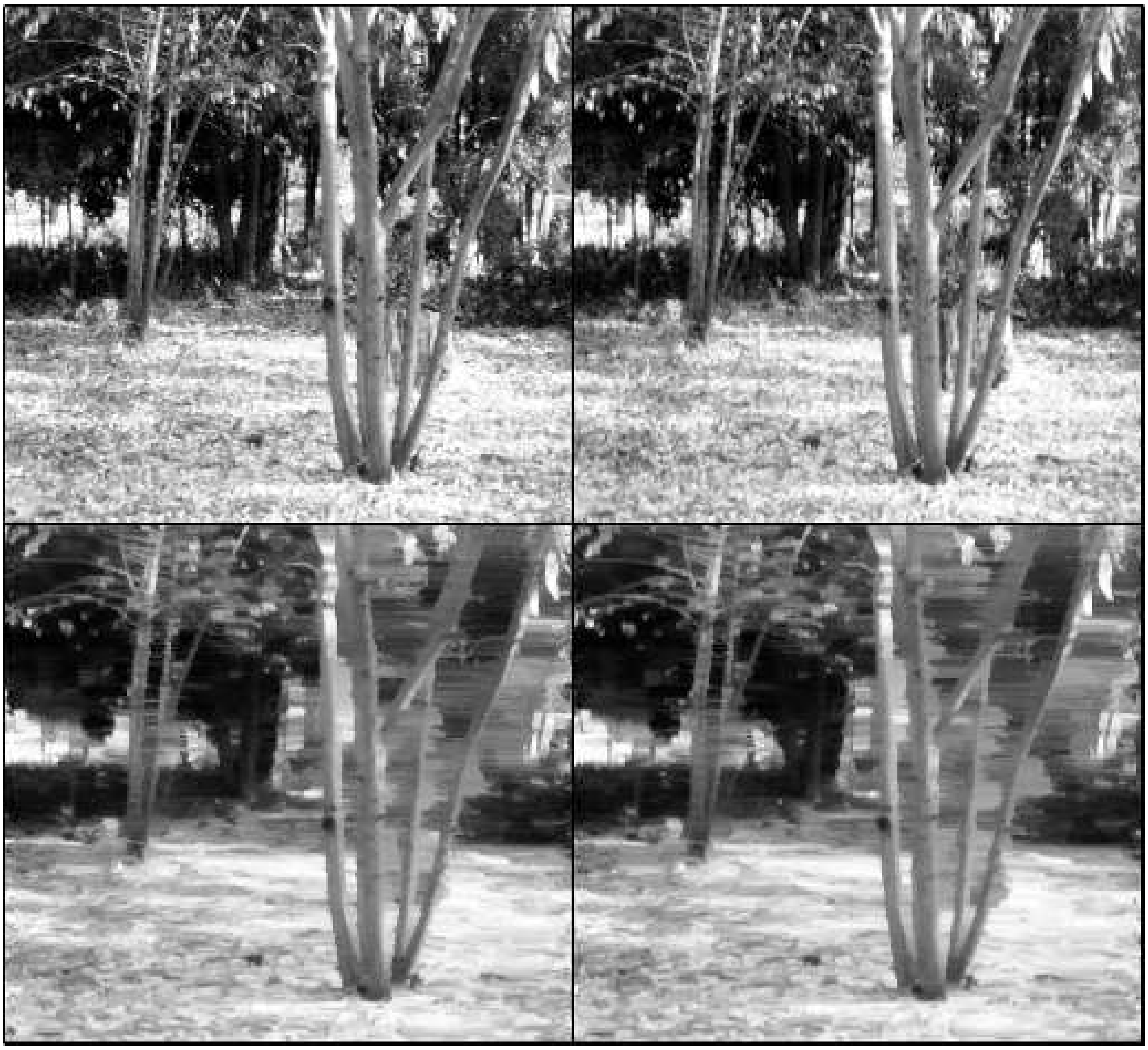

Figure 1: Filtering of the "TREES" movie.

*The two images on top row are images #18 and #22 (over a total of 64) of a real movie satisfying our assumptions on the camera motion (this movie was produced by the SRI center (see [2]), and is available by anonymous ftp to the adress* periscope.cs.umass.edu*). This movie was processed with 31 iterations of the above-mentionned inf-sup scheme of the DCMA, and on bottom row are images #18 and #22 of the processed movie. The original movie has small details which cannot be tracked between successive images (they are not time-coherent), because the Nyquist limit for the time frequencies has been exceeded during the sampling process. The strong smoothing effects of the analysis (on the ground for example) are necessary to ensure the time coherence of the movie. The smudging effects near the branches of the foreground tree, however, are undesired and due to the incapacity of the DCMA to handle occlusions.*

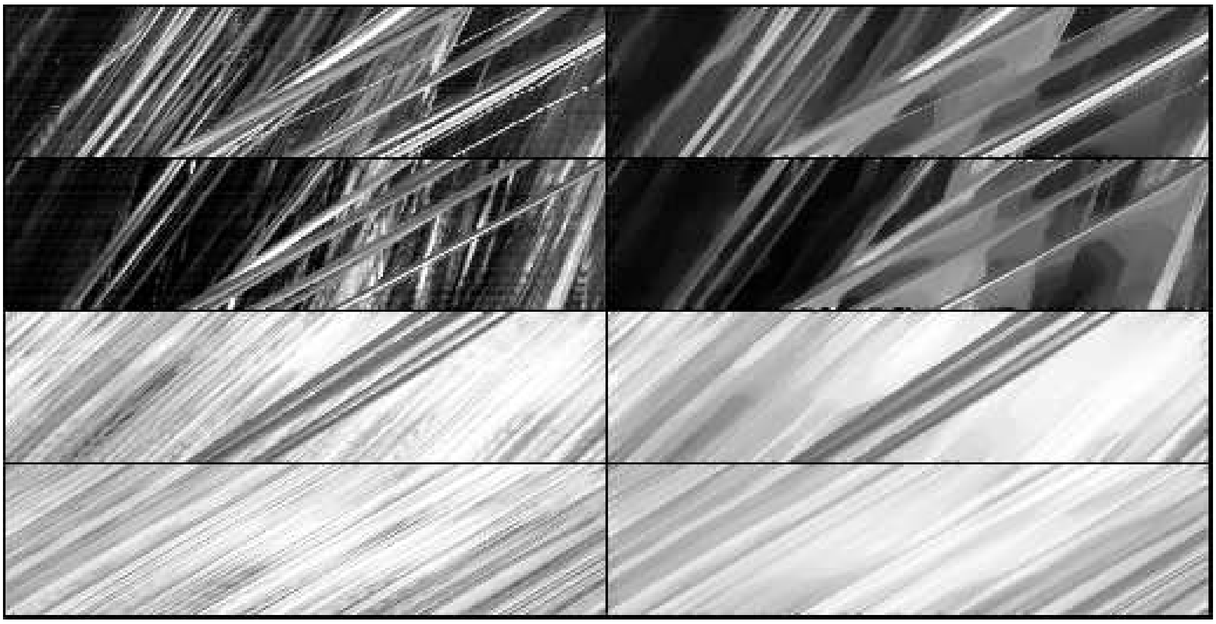

Figure 2: Analysis of the epipolar images.

*The epipolar images are obtained by slicing the movie* $u(x, y, \theta)$ *along* $(x, \theta)$ *planes for fixed values of* $y$*. The resulting images* $(x, \theta) \mapsto u(x, y, \theta)$ *are represented as follows : the* $x$ *axis is taken horizontal and the time axis* $\theta$ *is taken vertical pointing downwards. The epipolar images on colum 1 are taken from the original "TREES" movie (the values of* $y$ *are 20, 60, 180, 220 respectively for rows 1, 2, 3, 4, 5). Those on column 2 are obtained after processing the original ones with 31 iterations (the DCMA operates independently on all these epipolar images). One can notice that the time-coherence increase with scale, as the grey-level fluctuations vanish along trajectories as scale increases.*

*On the original epipolar images, occlusions appear when two lines intersect : only the one with the smallest slope (i.e. representing the object closest to the camera) remains during the occlusion, the other one being occluded. Notice that occluded objects are often destroyed by the DCMA (see row 2 for example), because the DCMA cannot handle occlusions.*

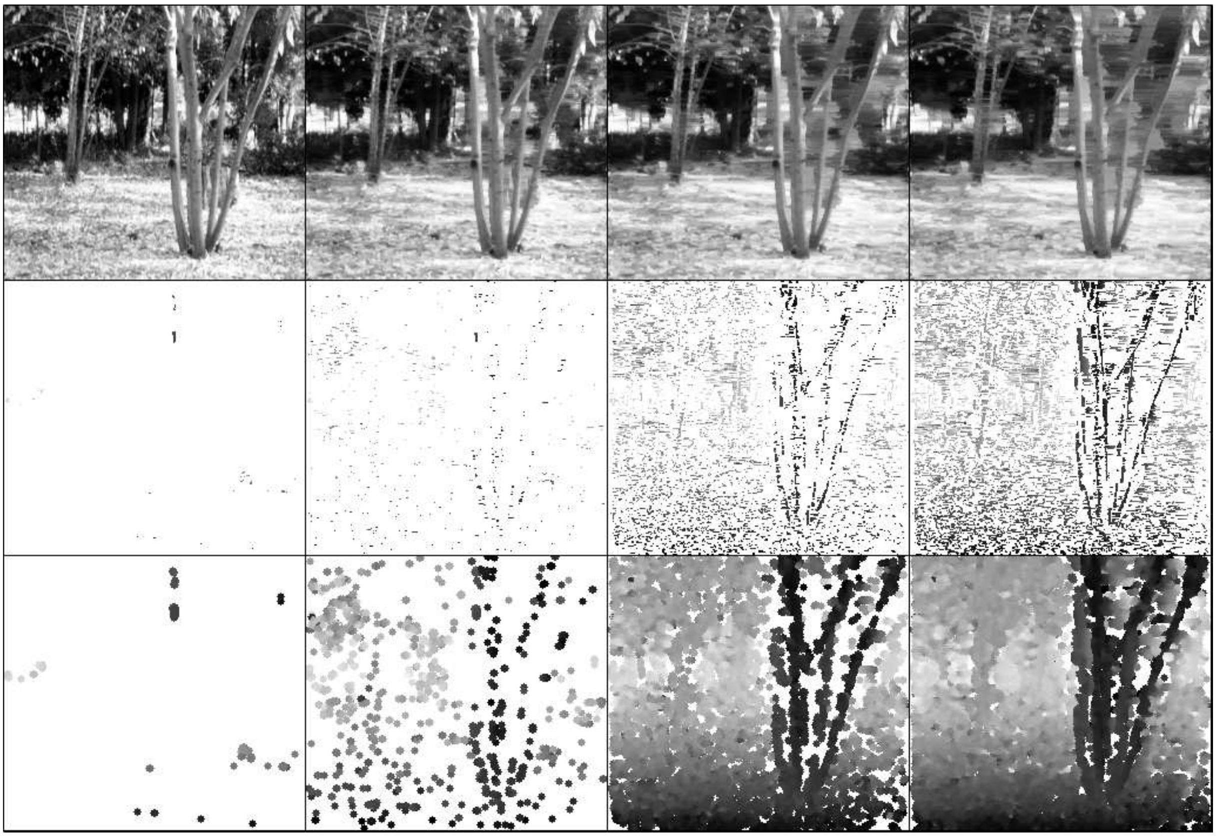

Figure 3: Computation of the velocity field (minimum of 15 matchings).

*The four images on the first row are taken from four different movies : each image is the 20th image (over 64) of the movie it belongs to : column 1: original "TREES" movie, column 2: processed movie (5 iterations), column 3: processed movie (15 iterations), column 4: processed movie (31 iterations).*

*Of course, since the DCMA is devoted to the depth recovery — or, equivalently, to the computation of the velocity field —, it would not be enough to show filtered movies without checking the consequences of the DCMA on their inherent velocity fields. For that reason, we need to devise an algorithm to compute such velocity fields. Now comes the interest of the DCMA : since it produces a perfect time-coherent movie, a naïve algorithm can be used to compute the velocity field.*

*Thus, the velocity field of each movie was computed on the 20th image simply by looking for trajectories with a matching constraint of 15 images. These velocities are represented on row 2 : the white color means points where no matching was found with respect to the constraint, and the grey scale (from light grey to black) measures the velocity from 0.0 to 2.0 pixels per image. On the third row, the velocity images of row 2 were "dilated" to produce more readable results. Notice how the velocity information, which is almost inexistant on the original movie (for the matching constraint we imposed), progressively appears on the DCMA as the scale increases. Since the distance of objects to the image plane is inversely proportional to their velocity, closest points appear in black and farthest ones in light grey. On the last image of row 3, we distinguish the foreground tree in black, the ground from black to middle grey, the background tree in middle grey, and the far background in light grey.*

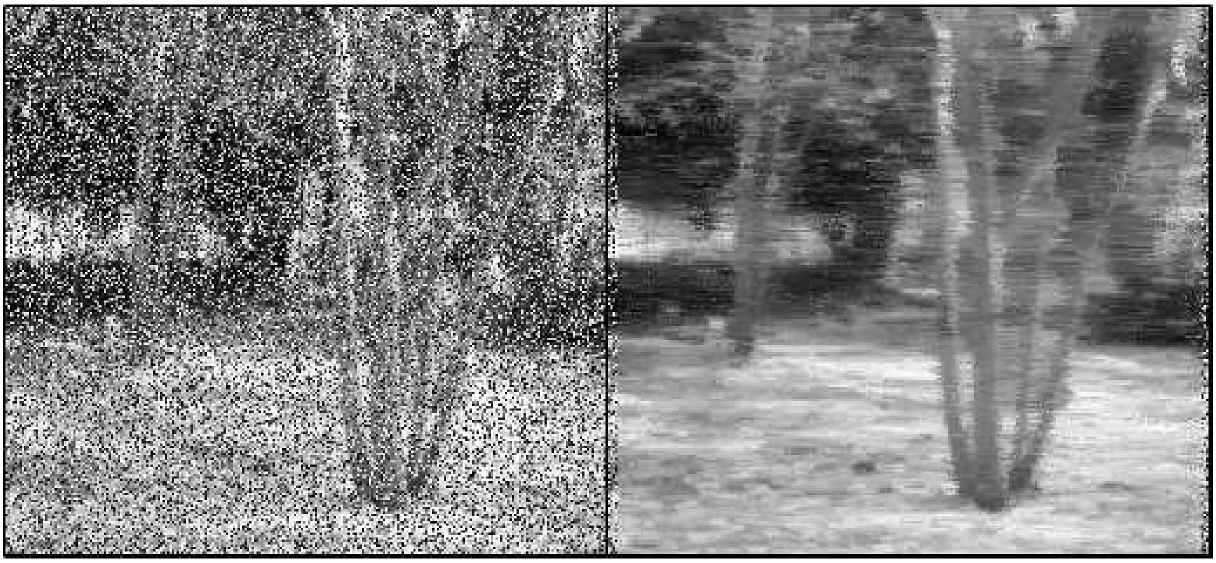

Figure 4: Filtering of the noisy "TREES" movie.

*The original "TREES" movie was corrupted with a severe impulse noise : 50% of the pixel values on each image were modified using a non-correlated, uniformly distributed random generator (and this 50% amount of pixel was chosen itself by a random generator). When playing the noisy movie movie (left : image #20), one has the impression of looking at a TV-image received in very poor conditions. In particular, it is almost impossible to see any detail of the ground texture. Filtering this movie with the DCMA (right : image #20 of the processed movie) gives interesting enhancement results : not only the noise impression is almost removed, but in addition some details appear that were not visible on the first movie (in particular on the ground and on the left tree). This means that the DCMA takes more advantage of the time coherence and redundancy contained in such a movie than the human visual system does.*

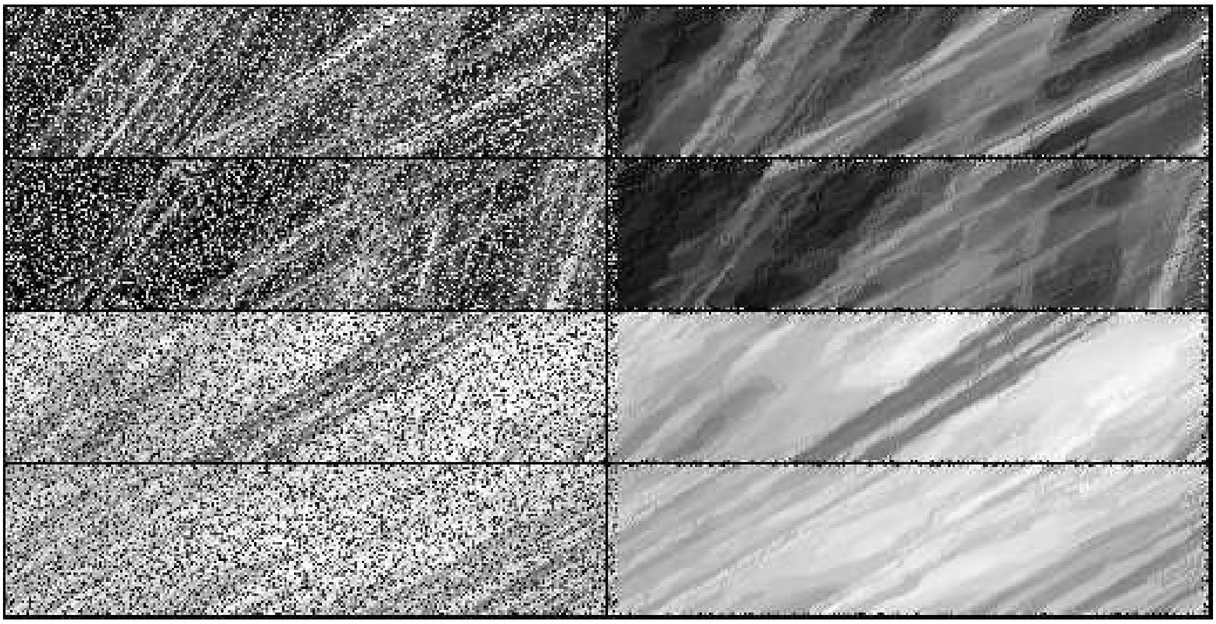

Figure 5: Analysis of the epipolar images.

*As on Figure 2, epipolar images are shown both for the original noisy movie (column 1) and for its processed version after 31 iterations (column 2).*

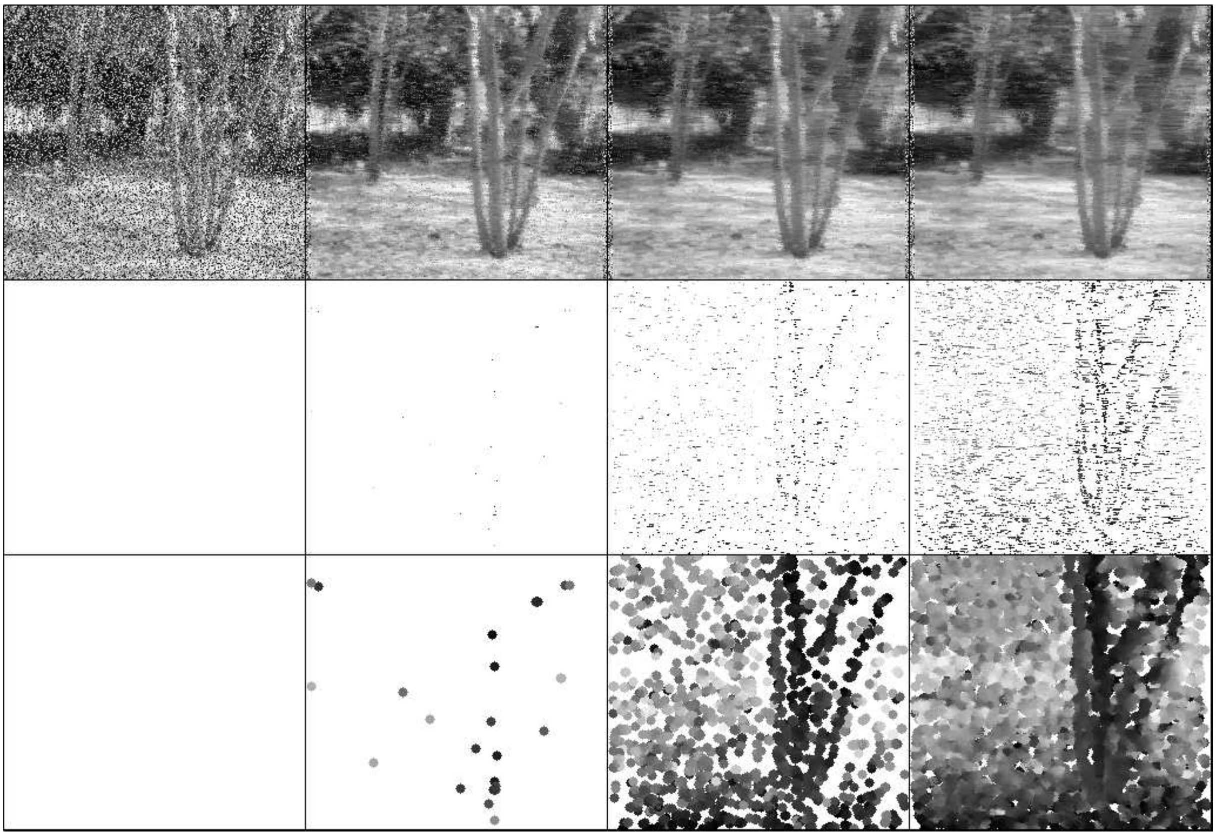

Figure 6: Computation of the velocity field (minimum of 15 matchings).

*Like on Figure 3, the four images on the first row are the 20th image of four different movies :*

| | |
|---|---|
| *column 1:* | *original noisy "TREES" movie* |
| *column 2:* | *processed movie (5 iterations)* |
| *column 3:* | *processed movie (15 iterations)* |
| *column 4:* | *processed movie (31 iterations)* |

*Row 2 and 3 represent the extracted velocity field (for a minimum of 15 matching images), in the original (row 2) and dilated (row 3) representation. As expected, not only the movie is filtered, but the velocity of objects is recovered despite a lot of destructed clues due to the large amount of noise put on the movie. Of course, the velocity recovery is not as good as if the movie had not been initially corrupted, but the depth structure of the scene still appear on the bottom-right image.*