Anisotropic Mumford-Shah Model
Workshop Imagerie Angiographie

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We want to perform the segmentation of the 3-D blood network. Segmentation of the biggest vessels are easily done by classical methods. The problem comes from the detection of thin tubular structures with section and intensity comparable to noise.
At the small scale of thin structures, the only criterium to discriminate vessels from noise is the tubular geometry. So, we want to define an energy which contains a geometric prior on the image we want to get.
At the small scale of thin structures, the only criterium to discriminate vessels from noise is the tubular geometry. So, we want to define an energy which contains a *geometric prior* on the image we want to get.

A first approach may consist in taking an energy defined as

\[ \mathcal{E} = \mathcal{E}_{\text{classical}} + \mathcal{E}_{\text{geometric}} \]

where we choose the first term as a well known energy (Rudin-Osher-Fatemi, Modica-Mortola, Mumford-Shah,...) and the second term is a distance from the given image to an geometric model.
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2) \( \uparrow \) complexity \( \Rightarrow \) \( \downarrow \) interpretation,
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has several drawbacks:

1) \( \uparrow \) complexity \( \Rightarrow \downarrow \) robustness,

2) \( \uparrow \) complexity \( \Rightarrow \downarrow \) interpretation,

3) the Classical Energy contains itself a geometric prior.

We rather incorporate the geometric prior \emph{inside} the classical energy.
For binary images, we proposed an energy based the classical Modica-Mortola model in such a way that the geometric prior is incorporated inside the energy.

Anisotropic Bimodal Energy for Segmentation of thin tubes and its approximation with $\Gamma$-convergence, Advances in Calculus of Variation, V., 2015

This work may also be done for a large class of energies. In this presentation we will present it in the context of the Mumford-Shah model.
Let $\Omega \subset \mathbb{R}^n$ be a domain and $g : \Omega \rightarrow \mathbb{R}$ an image, Mumford-Shah energy is defined as

$$E(u, K) = \int_{\Omega \setminus K} (u - g)^2 \, dx + \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(K).$$

where $K \subset \Omega$ is compact, and $u \in W^{1,2}(\Omega \setminus K)$. 
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where $K \subset \Omega$ is compact, and $u \in W^{1,2}(\Omega \setminus K)$.

We consider $g = \alpha 1_E$ and $u = g$, then

$$E(\alpha 1_E, \partial E) \leq E(0, \emptyset) \iff \mathcal{H}^{n-1}(\partial E) \leq \alpha \mathcal{L}^n(E).$$

This means that for a given intensity $\alpha$, the more the ratio $\frac{\mathcal{H}^{n-1}(\partial E)}{\mathcal{L}^n(E)}$ is small, the more the set $E$ is good for this energy.
The ratio $\frac{\mathcal{H}^{n-1}(\partial E)}{\mathcal{L}^n(E)}$ is optimal for the balls. The more a set $E$ has an isotropic geometry, the less is this ratio. So, if we force the Mumford-Shah model to detect thin structures, it is impossible to discriminate the noise from the tubes with same section and intensity.
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1. $\mathcal{L}^n(E)$ can not be changed because it comes from the data fitting term $\int_{\Omega \setminus K} (u - g)^2 \, dx$, 

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David Vicente

Anisotropic Mumford-Shah Model
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We have to modify the Mumford-Shah energy to favor anisotropic structures:

1. \( \mathcal{L}^n(E) \) can not be changed because it comes from the data fitting term \( \int_{\Omega\setminus K} (u - g)^2 \, dx \),

2. \( \mathcal{H}^{n-1}(\partial E) \) comes from \( \mathcal{H}^{n-1}(K) \), it is isotropic, we have to change it.
We will transform $\mathcal{H}^{n-1}(K)$ in such a way to incorporate a preference for sets *having a direction*. Our idea is to associate a new metric which is elongated in one direction (*anisotropic*). Moreover, this metric depends on the point of the domain (*inhomogeneous*).
For a $C^1$-surface $K$, we replace $\mathcal{H}^{n-1}(K)$ by

$$\int_K \langle \mathbf{M} \nu, \nu \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

- $\nu$ is an unitary and orthogonal vector to $K$,
- $\mathbf{M}$ is a given riemannian metric, that is $\mathbf{M} : \Omega \to S_n^+$. 
For a $C^1$-surface $K$, we replace $\mathcal{H}^{n-1}(K)$ by

$$\int_K \langle Mv, v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

- $v$ is an unitary and orthogonal vector to $K$,
- $M$ is a given riemannian metric, that is $M : \Omega \rightarrow S_n^+.$

The associated anisotropic energy is

$$\mathcal{E}_M(u, K) = \int_{\Omega \setminus K} (u - g)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_K \langle Mv, v \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1},$$

where $K$ is a compact $C^1$-surface and $u \in W^{1,2}(\Omega \setminus K).$
In this talk we will give answers to the questions:

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3) How construct the metric $M$?
To prove existence of a solution for the associated minimizing problem, we introduce a relaxation of the model.
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i) $u \in L^1(\Omega)$,

ii) $Du$ is a Radon measure,

iii) $Du = \nabla u \cdot dx + (u^+ - u^-) \cdot d\mathcal{H}^{n-1} \mathcal{L} J_u$. 

11/1

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$SBV(\Omega)$ is a Banach space and $J_u$ has the following structure.

**Theorem**

If $u \in SBV(\Omega)$, then $J_u$ is $\mathcal{H}^{n-1}$-rectifiable, i.e. $J_u = N \cup C$ with

- $\mathcal{H}^{n-1}(N) = 0$,

- $C = \bigcup_{i=1}^{\infty} \Gamma_i$ is a countable union of $C^1$ compact surface.
We give the relaxed formulation
\[
\mathcal{E}_M(u, K) = \int_{\Omega \setminus K} (u - g)^2 \, dx + \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_K \langle M\nu, \nu \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1}.
\]

\[
\downarrow
\]

\[
\mathcal{E}_M^r(u) = \int_{\Omega} (u - g)^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx + \int_{J_u} \langle M\nu_u, \nu_u \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1},
\]

where \( u \in SBV(\Omega) \) and \( \nu_u \) is unitary and orthogonal to \( J_u \).
We use the *direct method* of calculus of variations.

**Théorème [Ambrosio, ’95, lower semicontinuity]**

If \((u_k)_k \subset SBV(\Omega)\) satisfies

\[
\sup_k \left\{ \int_\Omega |\nabla u_k|^2 \text{d}x + \mathcal{H}^{n-1}(J_{u_k}) \right\} < \infty
\]

and \((u_k)_k\) converge \(*\)-weakly to \(u \in SBV(\Omega)\), then \((\nabla u_k)_k\) converge to \(\nabla u\) in \(L^1(\Omega)\) and

\[
\int_\Omega |\nabla u|^2 \text{d}x \leq \liminf_{k \to \infty} \int_\Omega |\nabla u_k|^2 \text{d}x,
\]

\[
\mathcal{H}^{n-1}(J_u) \leq \liminf_{k \to \infty} \mathcal{H}^{n-1}(J_{u_k}).
\]
Theorem [Ambrosio, ’95, compactness]

Let \((u_k)_k \subset SBV(\Omega)\) be as in the previous theorem. If \(\|u_k\|_\infty\) is uniformly bounded in \(k\), then there exists a subsequence of \((u_k)_k\) which converges \(\star\)-weakly to \(u \in SBV(\Omega)\).
Theorem [Ambrosio, ’95, compactness]

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If \(M\) satisfies

i) ellipticity:

\[
\exists \lambda > 0, \exists \Lambda > 0, \forall (x, v) \in \Omega \times \mathbb{R}^n, \quad \lambda |v|^2 \leq \langle M(x)v, v \rangle \leq \Lambda |v|^2,
\]

ii) Hölder-regularity:

\[
\exists \alpha > 0, \exists C \geq 0, \forall (x, y) \in \Omega^2, \quad \|M(x) - M(y)\| \leq C|x - y|^\alpha.
\]

then we adapt the two previous theorems and prove that there exists a minimizer \(u \in SBV(\Omega)\) of \(E^r_M\).
We have

\[ \text{Min} \mathcal{E}_M^r \leq \text{Min} \mathcal{E}_M, \]

\[ u \text{ is a minimizer of } \mathcal{E}_M^r \]

\[ \mathcal{H}^{n-1}(\overline{Ju} \setminus J_u) = 0 \]

\[ \Rightarrow (u, \overline{Ju}) \text{ is a minimizer of } \mathcal{E}_M. \]
In general $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$ is not true. For example, we may construct a function $SBV$ such that $J_u = S$ and $\overline{J_u} = S \cup V$ as below.

To prove that a minimizer of the relaxed problem provides a minimizer of the initial problem, it remains to prove that this situation is impossible.
The bad situation for $J_u$ is due to the fact that the complexity of $S$ is increasing at its boundary. More precisely, we introduce the local entropy as follows

$$I_x(\rho) = \frac{1}{\rho^{n-1}} \left( \mathcal{H}^{n-1}(J_u \cap \overline{B_\rho(x)}) + \int_{B_\rho(x)} |\nabla u|^2 \, dx \right).$$
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We have the typical behavior

- if $x \in S$ then $I_x(\rho)$ is decreasing when $\rho \to 0$,
- if $x \in V$ then $I_x(\rho)$ tends to $+\infty$ when $\rho \to 0$. 
A minimizer \( u \) of the relaxed problem satisfies the following.

**Definition**

A function \( u \in SBV(\Omega) \) is said \((\Lambda, c)\)-quasi minimizer of a free boundary problem if for any \( x \in \Omega \) and \( v \in SBV(\Omega) \) such as \( \{ u \neq v \} \subset B_\rho(x) \), we have

\[
\int_{B_\rho(x)} |\nabla u|^2 dx + \mathcal{H}^{n-1}(J_u \cap \overline{B_\rho(x)}) \\
\leq \int_{B_\rho(x)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B_\rho(x)}) + c\rho^n.
\]
Theorem [Bucur, Luckhaus, ’14, Monotonicity formula]

Let $u \in SBV(\Omega)$ a $(\Lambda, c)$-quasi minimizer of a free boundary problem, then

$$\rho \rightarrow I(\rho) \wedge \frac{c\Lambda^{2-n}}{n-1} + (n-1)c\rho$$

is decreasing in a neighborhood of $0^+$.

This yields

Theorem

Let $u \in SBV(\Omega)$ a $(\Lambda, c)$-quasi minimizer of a free boundary problem, then $\mathcal{H}^{n-1}(J_u \setminus J_u) = 0$.

So, a solution of the relaxed problem provides a regular solution.
To give a formulation of this problem more suitable for numerics we replace the $C^1$ surface $K \subset \Omega$ by a smooth function $z : \Omega \to [0; 1]$. It is a sort of *diffusion* of the surface in the following sense:

$$z(x) \approx \begin{cases} 
1 & \text{if } x \text{ is close to } K, \\
0 & \text{otherwise},
\end{cases}$$
We approximate

$$\mathcal{E}_M(u,K) = \int_\Omega (u - g)^2 \, dx + \int_{\Omega \setminus K} |\nabla u|^2 \, dx + \int_K \langle Mv, v \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1},$$

by

$$\mathcal{E}_\varepsilon(u,z) = \int_\Omega \left( (u - g)^2 + |\nabla u|^2 (1 - z)^2 + \varepsilon \langle M\nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) \, dx.$$

This functional is defined on $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ and it is elliptic. So, it may be minimized by a non linear heat equation.
To justify this approximation, we prove the following result.

**Theorem (Γ-convergence)**

If \((\varepsilon_k)_k\) converges to \(0^+\) then the functionals \((\mathcal{E}_{\varepsilon_k})_k\) Γ-converges to \(\mathcal{E}_M\).

If \((u_k, z_k)\) is a minimizer of \(\mathcal{E}_{\varepsilon_k}\), then there exists a subsequence of \((u_k, z_k)_k\) which converges almost everywhere to \((u, 0)\) and \(u\) is a minimizer of \(\mathcal{E}_M\).
The proof of $\Gamma$-convergence is divided in two steps.

**Theorem (lim-inf of $\Gamma$-convergence)**

If $(\varepsilon_k)_k$ converges to $0^+$ and $(u_k, z_k)_k$ converges almost everywhere to $(u, 0)$, then

$$\liminf_{k \to \infty} \mathcal{E}_{\varepsilon_k}(u_k, z_k) \geq \mathcal{E}_M^r(u).$$

We reduce to dimension one by using a *Slicing Property* of SBV functions.

Approximation of Functionals Depending on Jumps by Elliptic Functionals via Gamma-Convergence, 1990, Ambrosio-Tortorelli
The second step of the proof consists in showing that the previous lower bound is optimal. For that, we recall the upper and lower Minkowski contents of a set.

\[
\mathcal{M}^{*}(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega : \text{dist}(x, S) < \rho\})}{2\rho},
\]

\[
\mathcal{M}_{*}(S) = \liminf_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega : \text{dist}(x, S) < \rho\})}{2\rho},
\]

when it coincides, we call it the Minkowski contents of a set \(\mathcal{M}(S)\).
We introduce an anisotropic and inhomogeneous version of the Minkowski content.

\[ \mathcal{M}_M^*(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{ x \in \Omega : \text{dist}_M(x, S) < \rho \})}{2\rho} \]

where dist\(_M\) is the distance associated to the dual metric of M.

**Theorem**

If \((\varepsilon_k)_k\) converges to 0\(^+\) and \(u \in SBV(\Omega)\), then there exists \((u_k, z_k)_k\) converging to \((u, 0)\) a.e. such as

\[ \limsup_{k \to \infty} \mathcal{E}_{\varepsilon_k}(u_k, z_k) \leq \int_\Omega (u - g)^2 dx + \int_\Omega |\nabla u|^2 dx + \mathcal{M}_M^*(J_u). \]
Theorem, Geometric Measure Theory, Federer

If $W$ is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then

$$\mathcal{M}(W) = \mathcal{H}^{n-1}(W).$$

We generalize this result to the global setting of an anisotropic and inhomogeneous metric as follows.

**Theorem**

If $M$ is elliptic and Hölder regular and if $W$ is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then

$$\mathcal{M}_M(W) = \int_W \langle M\nu, \nu \rangle d\mathcal{H}^{n-1},$$

where $\nu$ is an unitary and orthogonal vector to $W$. 
According to the Entropy Decay Property, we prove the following.

**Theorem**

If \( u \in SBV(\Omega) \) is a quasi minimizer of a free boundary problem then

\[
\mathcal{M}_M(J_u) = \int_{J_u} \langle M\nu_u, \nu_u \rangle d\mathcal{H}^{n-1},
\]

where \( \nu \) is an unitary and orthogonal vector to \( W \).

**Theorem, Upper \( \Gamma \)-limit**

For any \( u \in SBV(\Omega) \), there exists \((u_k, z_k)_k\) converging a.e. to \((u, 0)\) a.e.

\[
\limsup_{k \to \infty} E_{\varepsilon_k}(u_k, z_k) \leq \int_\Omega (u - g)^2 + \int_\Omega |\nabla u|^2 + \int_{J_u} \langle M\nu_u, \nu_u \rangle d\mathcal{H}^{n-1}.
\]
We give a definition adapted to dimension 2. For that, we search for an unitary vector field $c : \Omega \to S^1$ following the direction of the tubes.

**Figure**: Vector field $c$ along a tube $K$
We introduce
\[ F(c) = \int_{\Omega} \langle Dg, c \rangle^2 \, dx + \int_{\Omega} |Dc|^p \, dx \]

\[(P_c) : \min \{ F(c) : c(x) \in S^1 \text{ a.e. } x \in \Omega, c \in W^{1,p}(\Omega) \} .\]

If we set \( p > 2 \) then, Sobolev embedding Theorem ensures that \( c \) is \( \alpha \)-Holder regular with \( \alpha = 1 - \frac{2}{p} \). It is easy to prove that a solution \( c_0 \) of \((P_c)\) exists and we set
\[ M = \text{Id} + \mu^t c_0 c_0, \]

where \( \mu > 0 \) corresponds to the elongation of its unit ball.
Initial Image
Anisotropic Mumford-Shah Model
In dimension 3, the previous approach is not adapted. In fact, a vector field can avoid \textit{laterally} a tube without penalizing the regularization term $\int_{\Omega} |Dc|^p dx$.

\textbf{Figure}: Vector field $c$ avoiding lateraly a tube $K$
To overpass this problem, we introduce the second order derivative of $H$ of $g$ and the following minimization problem

$$F(M) = \int_\Omega \| M - H \|^2 dx + \int_\Omega \| DM \|^p dx$$

and the following minimization problem

$$(\mathcal{P}_H): \quad \min\{ F(M) : M \text{ satisfies Ellipticity condition, } M \in W^{1,p}(\Omega) \}.$$ 

If we assume that $H \in L^2(\Omega)$, then it easy to prove that this problem admits a solution $M_0$. As for the 2D case, we assume that $p > 3$ and Sobolev embedding Theorem ensures that $M_0$ satisfies Holder regularity.
To sum things up, this approach has been developed for:

- Modica-Mortola Energy,

[Anisotropic Bimodal Energy for Segmentation of thin tubes and its approximation with $\Gamma$-convergence, Advances in Calculus of Variation, V., 2015](#)

- Mumford-Shah Energy, (Thesis Manuscript (in writing), HAL-01132067 (for the first part of this talk)).

It may be extended to another variational problems. The principle is to include the geometric prior inside the regularization term of the energy.
Thank you for your attention!