Non-Cooperative Spectrum Access in Cognitive Radio Networks: a Game Theoretical Model

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Abstract

Cognitive radio networks provide the capability to share the wireless channel with licensed (primary) users in an opportunistic manner. Primary users have a license to operate in a certain spectrum band; their access can only be controlled by the Primary Operator and is not affected by any other unlicensed (secondary) user. On the other hand, secondary users (SUs) have no spectrum license, and they attempt to exploit the spectral gaps left free by primary users.

This work studies the spectrum access problem in cognitive radio networks from a game theoretical perspective. The problem is modeled as a non-cooperative spectrum access game where secondary users access simultaneously multiple spectrum bands left available by primary users, optimizing their objective function which takes into account the congestion level observed on the available spectrum bands.

As a key innovative feature with respect to existing works, we model accurately the interference between SUs, capturing the effect of spatial reuse. Furthermore, we consider both non-elastic and elastic user traffic, to model real-time as well as data transfer applications. Finally, we consider an alternative formulation of the spectrum access problem, where players use replicator dynamics to adjust their strategies, and we derive convergence conditions to Nash equilibrium points.

We demonstrate the existence of the Nash equilibrium, and derive equilibrium flow settings. Finally, we provide numerical results of the proposed spectrum access game in several cognitive radio scenarios, and study the impact of the interference between SUs on the game efficiency. Our results indicate that the congestion cost functions we propose in this paper lead to small gaps between Nash equilibria and optimal solutions in all the considered network scenarios, thus representing a starting point for designing pricing mechanisms so as to obtain a socially optimal use of the network.

Keywords:
Cognitive Radio Networks, Spectrum Access, Game Theory, Price of Anarchy, Elastic/Non-Elastic Traffic

1. Introduction

Even though the frequency spectrum is the scarcest resource for wireless communications, it results generally underutilized: in fact, actual spectrum usage measurements performed by the FCC's Spectrum Policy Task Force [1] reveal that at any given time and location, much of the prized spectrum lies idle. Such underutilization has stimulated a huge research effort in several domains (e.g., engineering, economics and regulation communities) to propose better spectrum management policies and techniques. For this reason, several dynamic spectrum access techniques have been recently proposed to better utilize the available spectrum, reducing its wastage.

Cognitive radio networks (CRNs) are envisioned to deliver high bandwidth to mobile users via heterogeneous wireless architectures and dynamic spectrum access techniques [2]. Such networks provide the capability to share the wireless channel with primary users in an opportunistic manner. In CRNs, a *primary* (or licensed) user has a license to operate in a certain spectrum band; his access is generally controlled by the Primary Operator (PO) and should not be affected by the operations of any other unlicensed user. On the other hand, unlicensed (*secondary*) users have no spectrum license, and they implement additional functionalities to share the licensed spectrum band without interfering with primary users.

In this work, we focus on the dynamic spectrum access problem in cognitive radio networks from a game theoretical perspective[1]. We consider multiple secondary users (SUs) competing in a non-cooperative way for a limited set of frequencies left available by primary users. As a consequence, game theory is the natural framework to study the interactions among such users.

Non-cooperative games for competitive spectrum access in cognitive radio networks have been recently considered in [4, 5, 6, 7, 8, 9, 10]. The works in [9, 10], which propose static and dynamic spectrum sharing schemes as well as spectrum pricing techniques, are somehow close to our work, but they do not model explicitly the interference between secondary users.

This paper overcomes this limitation by proposing a novel game theoretic model that solves the spectrum access problem in cognitive radio networks considering multiple POs and a given set of secondary users. More specifically, we consider a *non-cooperative spectrum access game* where secondary users access simultaneously multiple spectrum bands left available by primary users, optimizing their objective function which depends both on the total flow transmitted on each link (congestion cost) and the amount of flow that such user transmits on it.

The proposed game model is studied considering both non-elastic and elastic traffic demands (to model real-time as well as data transfer applications) that can be shipped over one or multiple frequency spectrum bands owned by different POs.

As a key innovative feature with respect to existing works, our spectrum access game models explicitly the interference between secondary users as well as the spatial reuse of

[1]Preliminary results of this work have been presented in [3].

frequencies. This is achieved introducing user-specific parameters that specify, for each available spectrum band, who are the interferers that contribute to the perceived link congestion.

We demonstrate the existence of the Nash Equilibrium Point (NEP), and derive equilibrium spectrum access settings. Furthermore, we perform a thorough numerical analysis of the proposed game in several CRN scenarios, measuring the efficiency of the equilibria of our game and discussing the causes that lead to efficiency loss. More in detail, we investigate systematically the impact of several parameters (like the number of SUs and wireless channels, as well as the interference between SUs) on the system performance, determining the *Price of Anarchy* (*PoA*) of the proposed spectrum access game. The *PoA* quantifies the loss of efficiency as the ratio between the cost of the worst Nash equilibrium and that of the optimal solution, which could be designed by a central authority. The *PoA*, therefore, indicates the maximum degradation due to distributed secondary users decisions (anarchy) [11].

We further model the dynamic spectrum access of SUs in cognitive radio networks as a *population* game [12] where players use replicator dynamics to adapt their strategies (i.e., the transmitted flows) to the congestion perceived on the available channels. We show that, under some conditions, the spectrum access game admits a potential function and replicator dynamics converges to *Lyapunov stable* Nash equilibrium points.

In summary, the numerical results indicate that:

- the congestion cost functions adopted in this paper lead to small gaps between Nash equilibria and optimal solutions in all the considered network scenarios, especially for non-elastic traffic, thus representing a starting point for designing pricing mechanisms that foster a socially optimal use of cognitive radio networks.
- The *PoA* depends significantly on the interference between SUs, and increases with both the number of SUs and that of wireless channels.
- For non-elastic traffic, the *PoA* is higher with partial interference between users than with full interference. An opposite result holds for elastic traffic, since in this case the solution which maximizes the social welfare is quite unfair, and hence far from the Nash Equilibrium solutions which generally provide more fair spectrum allocations.
- Under replicator dynamics, non-elastic SUs' flows always converge to the Nash equilibrium points of the spectrum access game.

The main contributions of this paper can therefore be summarized as follows:

- a novel non-cooperative spectrum access game where secondary users access simultaneously multiple spectrum bands left available by primary users, optimizing their objective function which depends on the total flow transmitted on a link (congestion cost) and the amount of flow that such user transmits on it. This game is studied taking into account the interference between SUs and considering both non-elastic and elastic traffic demands.

- The determination of the existence conditions for Nash equilibrium points.
- The proposition of an alternative formulation of the spectrum access game based on population games and replicator dynamics.
- A thorough performance analysis of the competitive spectrum access game under different system parameters.

The paper is structured as follows: Section 2 discusses related work. Section 3 introduces the network model, including users' objective functions, as well as the proposed cost functions. Section 4 demonstrates the existence of at least one Nash Equilibrium Point and illustrates a procedure to compute all equilibria for both non-elastic and elastic traffic demands. Section 5 illustrates an alternative formulation of the spectrum access game based on population games, and shows how replicator dynamics converges to Nash equilibrium points. Section 6 analyzes and discusses numerical results for the proposed model in several CRN scenarios. Finally, Section 7 concludes this paper.

2. Related Work

A survey on the different functionalities in cognitive radio networks and the related research challenges is presented in [13]. Specifically, the key issues related to spectrum management, spectrum mobility and spectrum sharing are discussed.

A game theoretical overview of dynamic spectrum sharing in cognitive networks is provided in [4], where several aspects are considered: analysis of network users' behaviors, efficient dynamic distributed design, and optimality analysis. Furthermore, an overview of different approaches to dynamic spectrum sharing is presented in [5].

In [6] the authors consider the following scenario: a set of K groups of users coexist in the same area, competing for the same unlicensed spectrum band. Each group consists of a single transmitter-receiver pair. The distributed spectrum access problem is modeled as a repeated game, and a self-enforcing truth-telling mechanism is proposed: if any greedy user deviates from cooperation, punishment will be triggered. Through Bayesian mechanism design, users have no incentive to reveal false channel conditions, and the competing users are enforced to cooperate with each other honestly.

A joint power/channel allocation scheme is proposed in [14] to improve the network performance using a distributed pricing approach. In this scheme, the spectrum allocation problem is modeled as a non-cooperative game, with each cognitive radio (CR) pair acting as a player. Each player is interested in maximizing his own achievable rate. A price-based iterative water-filling algorithm is proposed, which enables CR users to reach a good Nash equilibrium.

The previous work is extended in [7] by maximizing the sum-rate achieved by all cognitive radios over all channels. The coordinated channel access problem is formulated under a multi-level spectrum opportunity framework, and is solved with a centralized polynomial-time approximate algorithm.

An oligopoly pricing framework for dynamic spectrum allocation is presented in [8], in which primary users sell available spectrum to secondary users for monetary return. Furthermore, two approaches (called "strict constraints" and "QoS penalty") are presented to model primary users with limited resources. In the first approach, a low-complexity searching method is proposed to obtain the Nash Equilibrium and prove its uniqueness. In the "QoS penalty" based oligopoly model, a variable transformation method is developed to derive the unique Nash Equilibrium. Moreover, when the market information is limited, three myopically optimal algorithms are provided to enable price adjustment for duopoly primary users.

Both static and dynamic spectrum sharing schemes are introduced in [9] for a cognitive radio network consisting of one primary user and multiple secondary users sharing the same frequency spectrum. First, the authors model the spectrum sharing as an oligopoly market, and a static game is used to obtain the Nash equilibrium for the optimal allocated spectrum size for the secondary users. Then, the authors present a dynamic game in which each secondary user adapts his spectrum sharing strategy based on the observed marginal profit, which is a function of the spectrum price offered by the primary user. Finally, the performance of the proposed scheme is evaluated under different system parameters (e.g., radio channel quality, revenue function and learning rate for dynamic strategy adaptation).

In [10] the authors address the problem of spectrum pricing in a cognitive radio network where multiple primary operators compete with each other to offer spectrum access opportunities to the secondary users. Each of the primary operators aims to maximize its profit under quality of service constraints for primary users. Such situation is formulated as an oligopoly market consisting of a few firms and a consumer; secondary users are infinitesimally small, and they are represented by a utility function from which a demand function is derived. A Bertrand game model is considered and distributed algorithms are proposed to obtain the solution for this game.

These two previous works [9, 10] are quite similar to our problem; however, in [9], the authors consider only one primary operator and a set of secondary users, while it is more interesting to study the general case with multiple primary operators. Differently from our work, the authors do not take into account the interference between secondary users while expressing their objective functions. The work in [10] differs from ours by considering infinitesimally small secondary users: the authors assume that secondary users form a secondary service as a whole, and such service is represented by a given utility function and as a consequence by a demand function. However, in our work, we study a more general scenario, where multiple primary operators coexist and provide spectrum access to a finite set of secondary users characterized by non-elastic/elastic traffic demands.

Competitive routing games have been the focus of several works like those in [15, 16], where the network is shared by several users, each one having a non-negligible amount of flow to transmit. These two papers have presented conditions for the existence and uniqueness of the Nash equilibrium point considering general [15] and parallel links networks [16] with special link cost functions (polynomial or M|M|1).

We observe that the spectrum access game proposed in this paper extends the classical routing games considered in [15, 16], which represent a particular case of our game when full interference exists between all users. In fact, in our proposed game, the congestion cost perceived by each user depends both on the *set* of users that are transmitting on a given channel (link) and on the interference matrix. This feature captures the essence of spatial reuse in wireless systems in general (and in CRNs, in particular), and complicates consistently the analysis with respect to routing games. In fact, it has been demonstrated that the routing games studied in [15, 16] are characterized by a unique Nash equilibrium when polynomial cost functions (like those we consider in our work) are used. On the other hand, our spectrum access game is characterized by an infinite number of Nash equilibria, as we will show in Section 6.

3. The Interference-aware Spectrum Access Game

We consider a cognitive radio wireless system with a set $V = \{1, \ldots, N\}$ of Primary Operators (POs), each operating on a separate frequency spectrum, F_n, and having its own primary users, and a set $U = \{1, \ldots, I\}$ of secondary users (SUs), willing to share the frequency spectrums $\{F_1, \ldots, F_N\}$ with the primary users. The basic notation used in this paper is summarized in Table 1.

Table 1: Basic Notation

V	Set of Primary Operators (POs) (or frequency spectrum bands)
N	Number of available wireless channels ($N = \|V\|$)
U	Set of Secondary Users (SUs)
I	Number of SUs ($I = \|U\|$)
r^i	Traffic demand of non-elastic SU i
M^i	Upper bound on the demand of elastic SU i
m^i	Lower bound on the demand of elastic SU i
α_n^i	Utility value per flow unit of elastic SU i on channel n
g^i, h^i	Parameters of logarithmic utility of SU i
f_n^i	Flow transmitted by SU i on wireless channel n
f_n^{PU}	Total flow sent by primary users on channel n
f^i	Spectrum access strategy of SU i (i.e., flow vector of SU i)
f^{i*}	Optimal flow vector of SU i
f^{-i*}	Optimal flows of all SUs, except SU i
S^i	Spectrum access strategy space of SU i
S	Product strategy space
$F_{t,n}^i$	Total amount of flow observed by SU i on wireless channel n
A_n	Interference matrix on wireless channel n
$a_{i,k}^n$	Interference parameter between SU i and k on wireless channel n
a_n, b_n, β_n	Channel-specific cost parameters
$\lambda_n^i, \delta_i, \mu_n^i, \eta^i, \nu^i$	Lagrangian multipliers

Each SU can transmit simultaneously over multiple spectrum bands, splitting his traffic over the set of available channels, thus choosing which primary operators will transport his traffic. Two different traffic types are considered:

- Non-Elastic traffic: each SU $i \in U$ has a fixed amount of flow (r^i) to transmit, and aims at minimizing his objective function OF_N^i, which represents the total congestion cost perceived on all the used channels.

- Elastic traffic: in this case, users' demands are *elastic*, in the sense that they are function of the costs due to channel congestion, as well as of the utility perceived in transmitting the traffic over the available channels. Without loss of generality, we assume that each elastic SU is characterized by an upper bound (M^i) and a lower bound (m^i) on his demand, which represent his maximum and minimum traffic requirements, respectively. Therefore, in this case each SU $i \in U$ maximizes his degree of satisfaction (his objective function OF_E^i), which depends on both throughput (*utility*) and costs (*disutility*).

Let f_n^i denote the amount of flow that SU i sends on wireless channel n, and let f_n^{PU} be the total flow sent by primary users on such channel. The secondary user flow configuration $f^i = \{f_1^i, \ldots, f_N^i\}$ is called a spectrum access strategy of SU i, and the set of strategies $S^i = \{f^i \in R^N : f_n^i \geq 0, n \in V\}$ is called the spectrum access strategy space of SU i. The system flow configuration $f = \{f^1, \ldots, f^I\}$ is called a spectrum access *strategy profile* and takes values in the product strategy space S. Furthermore, let f^{-i} represent the flow configuration of all users except SU i.

We denote by A_n (which needs not be symmetric) the interference matrix associated with channel n, and by $a_{i,k}^n$, element of A_n, the interference parameter between secondary users i and k on wireless channel n. More specifically, $a_{i,k}^n$, $i, k \in U, n \in V$ is defined as follows:

$$a_{i,k}^n = \begin{cases} 1 & \text{if SU } i \text{ interferes with SU } k \text{ on channel } n \\ 0 & \text{otherwise} \end{cases}$$

Figure 1 illustrates an example scenario with one primary operator (PO_n) and 3 secondary users (SU_1, SU_2 and SU_3): SU_1 and SU_2 interfere with each other on channel n, while SU_3 does not interfere with any other user. Therefore, in this scenario the interference matrix A_n has the following form:

$$A_n = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that A_n can be also represented using an interference graph, which is still depicted in Figure 1.

In the following we present the objective functions of both elastic and non-elastic secondary users, as well as the cost functions we propose to adopt for wireless channels.

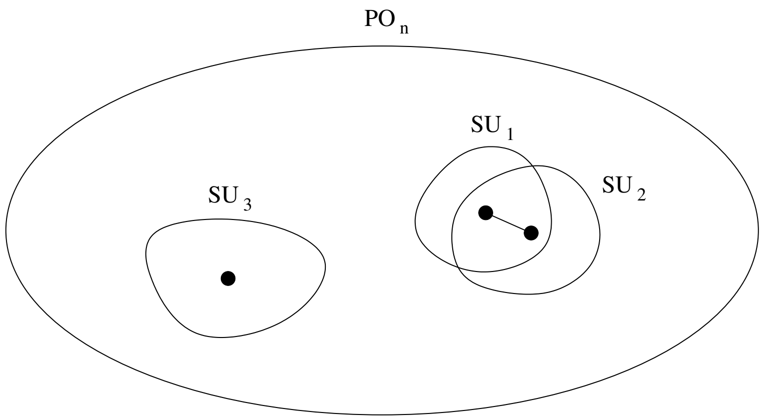


Figure 1: Example CRN scenario with one primary operator (PO_n) and 3 secondary users (SU_1, SU_2 and SU_3). SU_1 and SU_2 interfere with each other on channel n, while SU_3 does not interfere with any other user.

3.1. Non-Elastic Secondary User Objective Function

We associate to SU $i \in U$ the objective function OF_N^i, which is a function of the flow transmitted over the wireless channels:

$$OF_N^i(f^i, f^{-i}) = \sum_{n \in V} f_n^i \cdot J_n^i(f^i, f^{-i}). \tag{1}$$

The term $J_n^i(f^i, f^{-i})$ represents the cost on channel n per unit of flow, and OF_N^i is the total cost perceived by SU i over all available channels.

The solution concept adopted is the Nash Equilibrium, i.e., we seek a feasible multi-policy $f^* = f^{i*}, i \in U$ such that $OF_N^i(f^*) = OF_N^i(f^{i*}, f^{-i*}) = \min_{f_n^i} OF_N^i(f^i, f^{-i*})$, $i \in U$, where the minimum is taken over all policies f^i that lead to a feasible multi-policy together with f^{-i*}, which are the optimal flows of all secondary users $j \in U$ with $j \neq i$. Hence, each SU i aims to minimize his cost function OF_N^i:

$$\min_{f_n^i} \quad \sum_{n \in V} f_n^i \cdot J_n^i(f^i, f^{-i*}) \tag{2}$$

$$\text{s.t.} \quad \sum_{n \in V} f_n^i = r^i \qquad \forall i \in U \tag{3}$$

$$f_n^i \geq 0 \qquad \forall i \in U, n \in V. \tag{4}$$

3.2. Elastic Secondary User Objective Function

In this case, we associate to SU $i \in U$ the objective function OF_E^i:

$$OF_E^i(f^i, f^{-i}) = Q^i(f^i) - \sum_{n \in V} f_n^i \cdot J_n^i(f^i, f^{-i}). \tag{5}$$

The first term, $Q^i(f^i)$, represents the utility for transmitting a total amount of flow equal to $\sum_{n\in V} f_n^i$, and we assume that $Q^i(f^i)$ is continuous, concave and continuously differentiable in $f_n^i, \forall n \in V$. The second term, $\sum_{n\in V} f_n^i \cdot J_n^i(f^i, f^{-i})$, is the disutility due to the costs incurred for spectrum bands occupation.

As in the non-elastic case, for elastic SUs the solution concept adopted is the Nash Equilibrium, i.e., we consider a feasible multi-policy $f^* = f^{i*}, i \in U$ such that $OF_E^i(f^*) = OF_E^i(f^{i*}, f^{-i*}) = \max_{f_n^i} OF_E^i(f^i, f^{-i*}), i \in U$, where the maximum is taken over all policies f^i which lead to a feasible multi-policy together with f^{-i*}.

As a consequence, elastic SU i aims to maximize his objective function OF_E^i:

$$\max_{f_n^i} \quad \{Q^i(f^i) - \sum_{n\in V} f_n^i \cdot J_n^i(f^i, f^{-i*})\} \tag{6}$$

$$\text{s.t.} \quad m^i \le \sum_{n\in V} f_n^i \le M^i \qquad \forall i \in U \tag{7}$$

$$f_n^i \ge 0 \qquad \forall i \in U, n \in V. \tag{8}$$

3.3. Cost Function

In this work, we assume that the cost function is related to the total amount of flow that is transmitted on channel n. More specifically, the cost (or disutility) per flow unit perceived by SU i on channel n has the following form:

$$J_n^i(f) = a_n (F_{t,n}^i)^{\beta(n)} + b_n, \forall n \in V, \tag{9}$$

where $F_{t,n}^i = \sum_{k\in U} a_{k,i}^n f_n^k + f_n^{PU}$ is the total amount of flow observed by SU i over wireless channel n, taking into account the interference produced by all other secondary users as well as f_n^{PU}, which is the total amount of flow sent by primary users on wireless channel n. Parameters a_n, b_n and $\beta(n)$ are positive, and $\beta(n) \ge 1$. In this way, the cost perceived by SU i on channel n is polynomial (and hence convex) in the users' transmitted flows.

We will demonstrate that such cost function has appealing properties and ensures good Nash equilibria, which are, even in the worst cases, close to socially optimal solutions. We observe that, for $\beta(n) = 1$, $J_n^i(f)$ assumes an *affine* form; in this case, a possible interpretation for this type of secondary user cost in the context of telecommunication networks is that it is the expected delay of a packet in a light traffic regime [15]. Finally, this cost function can be also interpreted as a pricing function, which is used by the primary operator to charge the secondary users and gain some profit [9].

3.4. Utility Function

In the following, we will consider both affine and logarithmic utility functions for elastic Secondary Users. More specifically, we will consider SUs having the following utility $Q^i(f^i)$ in their objective function:

- (Affine utility) $Q^i(f^i) = \sum_{n\in V} \alpha_n^i f_n^i$, where α_n^i represents the utility value of SU i per unit of flow, on frequency spectrum band n.

- (Logarithmic utility) $Q^i(f^i) = g^i log(1 + h^i f^i)$, where g^i and h^i are the parameters of the logarithmic utility function of SU i. This is the same utility function used in [9, 17] to model elastic traffic. Note that with a logarithmic utility function, the rate of increase in utility $Q^i(f^i)$ decreases with increasing transmission rate f^i.

4. Existence and Computation of Nash Equilibria

Having defined our proposed interference-aware spectrum access game, in this section we first demonstrate that such game indeed admits at least a Nash equilibrium, and then we illustrate a procedure for computing its Nash Equilibrium Points (NEPs).

To this aim, we consider the cost function (9) and the utility functions introduced before. The objective function of SU $i \in U$ assumes therefore the following expression

$$OF^i_N(f^i, f^{-i}) = \sum_{n \in V} f^i_n [a_n (F^i_{t,n})^{\beta(n)} + b_n] \tag{10}$$

for non-elastic traffic and

$$OF^i_{EA}(f^i, f^{-i}) = \sum_{n \in V} \alpha^i_n f^i_n - \sum_{n \in V} f^i_n [a_n (F^i_{t,n})^{\beta(n)} + b_n] \tag{11}$$

$$OF^i_{EL}(f^i, f^{-i}) = g^i log(1 + h^i f^i) - \sum_{n \in V} f^i_n [a_n (F^i_{t,n})^{\beta(n)} + b_n] \tag{12}$$

for elastic traffic with *affine* (OF^i_{EA}) and *logarithmic* (OF^i_{EL}) utility function, respectively.

Non-elastic/elastic SUs' objective functions (10), (11) and (12) are continuous in $f = \{f^1, \ldots, f^I\}$ and, respectively, convex and concave in f^i_n. These properties ensure the existence of the Nash equilibrium according to the Kakutani fixed point theorem [18].

4.1. Computing the Nash equilibria for Non-Elastic Secondary Users

We now turn to the computation of the equilibrium solutions of our spectrum access game, starting from non-elastic secondary users.

In this case, each SU i aims at minimizing his objective function OF^i_N. By definition, a Nash equilibrium is the solution to the individual utility optimization problem for each user given all other users' actions. In our formulation, each individual optimization problem is a nonlinear convex problem with the linear constraints (3) and (4). So the Lagrangian function for user i can be written as:

$$\mathcal{L}^i_N(f^i, f^{-i}) = \sum_{n \in V} f^i_n [a_n (F^i_{t,n})^{\beta(n)} + b_n] - \sum_{n \in V} \lambda^i_n f^i_n + \delta^i (\sum_{n \in V} f^i_n - r^i) \tag{13}$$

where λ^i_n and δ^i are the Lagrangian multipliers (non negative real numbers). Based on nonlinear convex programming theory [19], the following Karush-Kuhn-Tucker (K.K.T.)

conditions are necessary and sufficient for a solution $f = \{f_n^i\}$ to be a Nash equilibrium:

$$a_n(F_{t,n}^i)^{\beta(n)} + a_n\beta(n)f_n^i(F_{t,n}^i)^{\beta(n)-1} + b_n = \lambda_n^i - \delta_i \quad \text{if } f_n^i > 0,\ \forall i \in U, n \in V \tag{14}$$

$$a_n(F_{t,n}^i)^{\beta(n)} + b_n \geq \lambda_n^i - \delta_i \quad \text{if } f_n^i = 0,\ \forall i \in U, n \in V \tag{15}$$

$$\sum_{n\in V} f_n^i = r^i \quad \forall i \in U \tag{16}$$

$$f_n^i \geq 0, \lambda_n^i \geq 0, \delta^i \geq 0 \quad \forall i \in U, n \in V. \tag{17}$$

4.2. Computing the Nash equilibria for Elastic Secondary Users

We first consider elastic SUs characterized by affine utility functions, and then move to logarithmic utility functions.

Each elastic SU i with an affine utility function aims at maximizing his objective function OF_{EA}^i. In this case, the Lagrangian function of the $i-$th user is the following:

$$\mathcal{L}_{EA}^i(f^i, f^{-i}) = \sum_{n\in V} \alpha_n^i f_n^i - \sum_{n\in V} f_n^i[a_n(F_{t,n}^i)^{\beta(n)} + b_n] + \\ + \sum_{n\in V} \mu_n^i f_n^i + \eta^i(\sum_{n\in V} f_n^i - m^i) - \nu^i(\sum_{n\in V} f_n^i - M^i) \tag{18}$$

where μ_n^i, η^i and ν^i are the Lagrangian multipliers. The K.K.T. conditions (19)-(22) reported hereafter give the Nash equilibrium solution.

$$\alpha_n^i - a_n(F_{t,n}^i)^{\beta(n)} - a_n\beta(n)f_n^i(F_{t,n}^i)^{\beta(n)-1} - b_n + \mu_n^i + \eta^i - \nu^i = 0 \quad \forall i \in U, n \in V \tag{19}$$

$$f_n^i\mu_n^i = 0,\ \eta^i(\sum_{n\in V} f_n^i - m^i) = 0,\ \nu^i(\sum_{n\in V} f_n^i - M^i) = 0 \quad \forall i \in U, n \in V \tag{20}$$

$$f_n^i \geq 0,\ \sum_{n\in V} f_n^i \geq m^i,\ \sum_{n\in V} f_n^i \leq M^i \quad \forall i \in U, n \in V \tag{21}$$

$$\mu_n^i \geq 0, \eta^i \geq 0, \nu^i \geq 0 \quad \forall i \in U, n \in V. \tag{22}$$

As for elastic SU i with a logarithmic utility function, he aims at maximizing his objective function OF_{EL}^i.

The Lagrangian function of the $i-$th user is in this case the following:

$$\mathcal{L}_{EL}^i(f^i, f^{-i}) = g^i log(1 + h^i f^i) - \sum_{n\in V} f_n^i[a_n(F_{t,n}^i)^{\beta(n)} + b_n] + \\ + \sum_{n\in V} \mu_n^i f_n^i + \eta^i(\sum_{n\in V} f_n^i - m^i) - \nu^i(\sum_{n\in V} f_n^i - M^i) \tag{23}$$

where μ_n^i, η^i and ν^i are the Lagrangian multipliers. The K.K.T. conditions (24)-(27) reported hereafter give the Nash equilibrium solution.

$$\frac{g^i h^i}{1 + h^i f^i} - a_n (F_{t,n}^i)^{\beta(n)} - a_n \beta(n) f_n^i (F_{t,n}^i)^{\beta(n)-1} - b_n + \mu_n^i + \eta^i - \nu^i = 0 \quad \forall i \in U, n \in V \tag{24}$$

$$f_n^i \mu_n^i = 0,\ \eta^i(\sum_{n \in V} f_n^i - m^i) = 0,\ \nu^i(\sum_{n \in V} f_n^i - M^i) = 0 \quad \forall i \in U, n \in V \tag{25}$$

$$f_n^i \geq 0,\ \sum_{n \in V} f_n^i \geq m^i,\ \sum_{n \in V} f_n^i \leq M^i \quad \forall i \in U, n \in V \tag{26}$$

$$\mu_n^i \geq 0, \eta^i \geq 0, \nu^i \geq 0 \quad \forall i \in U, n \in V. \tag{27}$$

4.3. Comments

As we will show in Section 6, our game can admit infinite Nash equilibria for both non-elastic and elastic traffic (i.e., the systems (14)-(17), (19)-(22) and (24)-(27) can have infinite solutions). Therefore, in the case of non-elastic traffic, to determine the highest-cost Nash equilibrium necessary to compute the Price of Anarchy, we further look for the feasible solution of (14)-(17) that maximizes the sum of all users' costs, $\sum_{i \in U} OF_N^i$. On the other hand, to compute the Price of Anarchy for elastic traffic, we look for the feasible solution of (19)-(22), for affine utilities, or (24)-(27), for logarithmic utilities, that minimizes the sum of all users' utilities (i.e., $\sum_{i \in U} OF_{EA}^i$ and $\sum_{i \in U} OF_{EL}^i$, respectively).

The SNOPT 7.2 solver [20], a software package for solving large-scale optimization problems, has been used to this end.

5. Replicator Dynamics for the Spectrum Access Game

In this section, we propose an alternative formulation of the dynamic spectrum access game, which we model as a *population game* [12] where the *replicator dynamics* is used to model the evolution of Secondary Users' strategies. This allows us to study a situation in which Secondary Users adjust their strategies, adapting their choices dynamically to the congestion perceived on each wireless channel.

To this aim, we first provide a background on population games, which represent a simple framework for describing strategic interactions among a large number of players. Then, we briefly illustrate the *replicator dynamics*, which permits to model the behavior of the agents that play such games.

5.1. Population Games

A population game G, with Q non-atomic classes of players is defined by a mass and a strategy set for each class, and a payoff function for each strategy. By a *non-atomic* *population* of players, we mean that the contribution of each member of the population is infinitesimal. The set of classes is denoted by $\mathcal{Q} = \{1, \ldots, Q\}$, and each class has associated a mass denoted by $m^q > 0$. Let S^q be the set of strategies available for players of class q, where $S^q = \{1, \ldots, s^q\}$.

During the game play, each player of class q selects a strategy from S^q. The mass of players of class q that choose the strategy $n \in S^q$ is denoted by x_n^q, where $\sum_{n \in S^q} x_n^q = m^q$. We denote the vector of strategy distributions being used by the entire population by $x = \{x^1, \ldots, x^Q\}$, where $x^i = \{x_1^i, \ldots, x_{s^i}^i\}$. The vector x can be thought of as the state of the system.

The marginal payoff function (per mass unit) of players of class q who play strategy n when the state of the system is x is denoted by $F_n^q(x)$, usually referred to as *fitness* in evolutionary game theory, which is assumed to be continuous and differentiable. The total payoff of the players of class q is therefore $\sum_{n \in S^q} F_n^q(x) x_n^q$.

5.2. Replicator Dynamics

The replicator dynamics describes the behavior of a large population of agents who are randomly matched to play normal form games. It was first introduced in biology by Taylor and Jonker [21] to model the evolution of species, and it is also used in the economics field. Recently, such dynamics has been applied to many networking problems, like routing and resource allocation [22, 23].

Given x_n^q, which represents the proportion of players of class q that choose strategy n, as illustrated before, the replicator dynamics can be expressed as follows:

$$\dot{x}_n^q = x_n^q \Big(F_n^q(x) - \frac{1}{m^q} \sum_{n \in S^q} F_n^q(x) x_n^q \Big),$$

where $\dot{x}_n^q$ represents the derivative of x_n^q with respect to time.

In fact, the ratio $\dot{x}_n^q / x_n^q$ measures the evolutionary success (the rate of increase) of a strategy n. This ratio can be also expressed as the difference in fitness $F_n^q(x)$ of the strategy n and the average fitness $\frac{1}{m^q} \sum_{n \in S^q} F_n^q(x) x_n^q$ of the class q.

5.3. Summary of results related to Replicator Dynamics

We now summarize the most notable results for the replicator dynamics (derived from [22, 24]), which help establishing the convergence of such dynamics to stable (Nash equilibrium) points.

Definition 1 The dynamics $\dot{x} = V(x)$ is said to be *positive correlated* (PC) if $\sum_{q \in Q} \sum_{n \in S^q} F_n^q(x) V_n^q(x) > 0$, whenever $V(x) \neq 0$.

Definition 2 A function $\Phi : X \to R$ is a *potential* for a game G if for every $i \in U$ and for every $x^{-i} \in X^{-i}$

$$\Phi(x, x^{-i}) - \Phi(z, x^{-i}) = u^i(x, x^{-i}) - u^i(z, x^{-i}), \forall x, z \in X^i,$$

where u^i represents the objective function (utility/cost) of user i.

G is called a *potential game* if it admits a potential function Φ [25].

Result 1 If V(x) satisfies PC, all equilibria of G are stationary points of $\dot{x} = V(x)$.

Result 2 The replicator dynamics is PC.

Result 3 A potential game G, with dynamics $V(x)$ that is PC, has asymptotically stable stationary points.

5.4. Replicator Dynamics for the Spectrum Access Game

We now use replicator dynamics to model and analyze the behavior of Secondary Users in the spectrum access game, focusing on *non-elastic* traffic.

In such formulation, a non-elastic SU i with traffic demand r^i can be considered as a class of population i with a global mass r^i of infinitesimal users; this is in line with the assumption made in [23]. The proportion of population i who uses strategy n (wireless channel n) is $x_n^i = \frac{f_n^i}{r^i}$.

The cost perceived by infinitesimally small SUs in population i who use channel n is given by:

$$C_n^i(x_n) = (x_n^i r^i)[a_n(\sum_{k \in U} a_{k,i}^n x_n^k r^k + f_n^{PU})^{\beta(n)} + b_n], \tag{28}$$

where $x_n = \{x_n^1, \dots, x_n^I\}$.

The replicator dynamics is then given by:

$$\dot{x_n^i} = x_n^i [V^i(x) - v_n^i(x_n)], \tag{29}$$

where $v_n^i(x_n)$ is the marginal congestion cost, which assumes the following form:

$$v_n^i(x_n) = \frac{\partial C_n^i}{\partial f_n^i} = a_n(\sum_{k \in U} a_{k,i}^n x_n^k r^k + f_n^{PU})^{\beta(n)} + a_n \beta(n) x_n^i r^i (\sum_{k \in U} a_{k,i}^n x_n^k r^k + f_n^{PU})^{\beta(n)-1} + b_n, \tag{30}$$

while $V^i(x) = \sum_{n \in V} x_n^i v_n^i(x_n)$ is the mean population cost.

The rationale behind equation (29) is the following: when player i observes a (marginal) congestion cost on channel n which is higher than the average cost experienced on all available channels, he decreases the amount of flow transmitted on such channel. In the opposite case, he reacts by increasing such amount.

If we consider the special case with $\beta(n)$=1 and $f_n^{PU} = 0, \forall n \in V$, then $v_n^i(x_n)$ becomes:

$$v_n^i(x_n) = a_n \sum_{k \in U} a_{k,i}^n x_n^k r^k + a_n r^i x_n^i + b_n, \tag{31}$$

and the following proposition holds.

Proposition 1. *Our proposed spectrum access game admits a potential function* Φ *which is given by the following expression:*

$$\Phi = \frac{1}{2}\sum_{i \in U} OF_N^i + \frac{1}{2}\Big(\sum_{i \in U}\sum_{n \in V}[a_n(f_n^i)^2 + b_n f_n^i]\Big). \quad (32)$$

Proof: See Appendix A.

Since we demonstrated that the proposed spectrum access game is a potential game, and knowing that replicator dynamics is positive correlated (see Result 2 in Section 5.3), we can conclude that:

- all Nash equilibria of the spectrum access game are rest points of the replicator dynamics, and
- the potential function Φ (32) acts as a *Lyapunov* function [24, 26] for $\dot{x_n^i} = x_n^i[V^i(x) - v_n^i(x_n)]$, and as a consequence, Nash equilibria of the spectrum access game are Lyapunov stable.

We will provide numerical examples that illustrate the convergence of the replicator dynamics towards Nash equilibrium points in Section 6.5.

6. Numerical Results

We now measure the sensitivity of the proposed spectrum access game to different parameters like the number of secondary users and wireless channels, the interference between SUs as well as the traffic demands. Furthermore, we study the efficiency of the Nash equilibria by comparing them to the socially optimal solutions, through the determination of bounds to the Price of Anarchy (PoA). Socially optimal solutions for *non-elastic* users minimize the sum of all users' costs, i.e., they minimize $\sum_{i \in U} OF_N^i$ subject to constraints (3)-(4); the PoA is defined as the ratio between the cost of the worst Nash equilibrium and that of the socially optimal solution. On the other hand, socially optimal solutions for *elastic* users maximize the sum of all users' utilities, i.e., they maximize $\sum_{i \in U} OF_{EA}^i$ subject to constraints (7)-(8), for affine utility functions, while they maximize $\sum_{i \in U} OF_{EL}^i$ subject to the same constraints (7)-(8), for logarithmic utility functions; in this case, the PoA is defined as the ratio between the utility of the socially optimal solution and that of the worst Nash equilibrium.

Several CRN scenarios have been considered. Some, very simple, have been studied to discuss preliminarily the main features of our proposed game. Then, more realistic random topologies with a large number of users and wireless channels are used to investigate the system performance.

All the results reported hereafter are the Nash equilibria and optimal solutions of the considered scenarios obtained, respectively, by formalizing the spectrum access models in AMPL, a modeling language for mathematical programming [27], and solving them with SNOPT 7.2 [20]. The computing time needed to solve each network instance was very small in all the considered scenarios, and in all cases it was smaller than 5 seconds.

However, in average, only fractions of a second are needed to solve our proposed model to the optimum.

In the following we discuss the numerical results obtained with an interference matrix A_n that is both symmetric and identical for all frequencies ($A_n = A_m, \forall n, m \in V$). If not specified differently, we consider cognitive radio scenarios with affine cost functions ($\beta(n) = 1, \forall n \in V$), and we set the cost parameters as follows: $a_n = 1, f_n^{PU} = b_n = 0, \forall n \in V$.

Obviously, our proposed model is general and can be applied also to asymmetric instances and with any parameters setting.

6.1. Non-Elastic Secondary Users: Simple CRN Scenarios

Hereafter we consider some simple Cognitive Radio Network topologies with different interference patterns, namely a 4-user and a chain network scenarios, discussing the quality of the Nash equilibria reached by secondary users through the determination of bounds to the Price of Anarchy.

6.1.1. 4-User CRN Scenarios

We first consider a cognitive radio network with two primary operators (2 wireless channels) and 4 secondary users (SU_1, SU_2, SU_3 and SU_4), all having the same traffic demand $r^i = 1$. To evaluate the impact of the interference matrix on the efficiency of our spectrum access game, we study 2 different scenarios: full interference (see Figure 2(a)) and partial (cyclic-like) interference (Figure 2(b)) between the SUs.

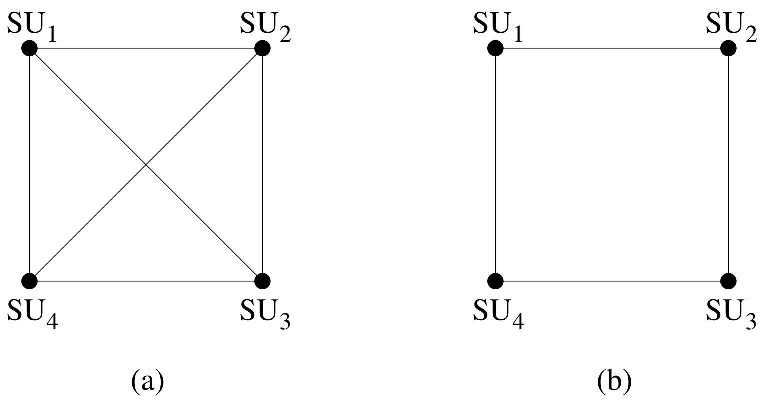


Figure 2: 4-User CRN scenarios: interference graphs. (a) Full interference and (b) partial (cyclic-like) interference between the SUs.

In the full interference case, the PoA is equal to 1, since both in the Nash equilibrium and in the optimal solution all SUs split their traffic equally on the available channels. On the other hand, in the partial interference scenario, there exist infinite Nash equilibrium points, i.e. those where SU_1 and SU_3 transmit p and $1-p$ traffic units on channels 1 and 2, respectively, while SU_2 and SU_4 transmit $1-p$ and p traffic units on these channels ($p \in [0, 1]$). The total cost of these equilibria is equal to $-8p^2 + 8p + 4$, which assumes its maximum value, 6, for $p = 1/2$; on the other hand, at the optimum, SU_1-SU_3 send all their traffic on one channel, and SU_2-SU_4 on the other, with a total cost of 4, thus leading to a $PoA = 3/2$.

Hence, it can be observed that the quality of the equilibria reached by cognitive radio users depends significantly on the specific interference scenario, and counter-intuitively, with full interference, we obtain a PoA which is lower than that obtained for the partial interference scenario.

If we consider quadratic cost functions (i.e., $\beta(n) = 2, \forall n \in V$), the PoA is equal to 1 in the full interference scenario, exactly as in the $\beta(n) = 1$ case discussed above; in the partial interference scenario, on the other hand, at least 3 equilibria exist, i.e. those where SU_1 and SU_3 transmit p and $1 - p$ traffic units on channels 1 and 2, respectively, while SU_2 and SU_4 transmit $1-p$ and p traffic units on these channels, with p equal to 0, 1 or 1/2. The first two equilibria ($p = 0$ and $p = 1$) coincide with the social optimum, which has an overall cost of 4, while for $p = 1/2$ the total cost of the Nash equilibrium point is 9, leading to a $PoA = 9/4$. Hence, increasing $\beta(n)$ leads to a higher Price of Anarchy, as we will quantify more precisely in Section 6.2.1.

Finally, we consider a variation of the $\beta(n) = 1$ scenario, where SU_1 and SU_3 have a rate r^i equal to 3/2 while the other two users have $r^i = 1/2$. As for the homogeneous traffic case, in the full interference scenario we again have $PoA = 1$, while for the partial interference scenario we have an infinite number of Nash equilibria: SU_1 and SU_3 transmit p and $3/2 - p$ traffic units on channels 1 and 2, respectively, while SU_2 and SU_4 transmit $1 - p$ and $p - 1/2$ traffic units on these channels ($p \in [1/2, 1]$). The total cost at the NEP is in this case equal to $-8p^2 + 12p + 1$, which is maximum for $p = 3/4$, where its value is 5.5. At the optimum, SU_1 and SU_3 send 5/4 and 1/4 traffic units on channels 1 and 2, respectively, while SU_2 and SU_4 send all their traffic on channel 2, with a social cost of 4.75, thus leading to a $PoA = 5.5/4.75 = 1.158$, which is lower than the one determined in the homogeneous traffic case (i.e., 3/2).

This result confirms on the one hand the behavior already observed for the homogeneous traffic case, where the PoA with partial interference is higher than the one with full interference. However, this effect is mitigated by the presence of heterogeneous traffic demands, as we will discuss more in detail for random CRN scenarios.

6.1.2. Chain-like Interference CRN Scenario

We now consider a chain-like interference scenario, illustrated in Figure 3, with 2 wireless channels and I secondary users. All users have $r^i = 1$.

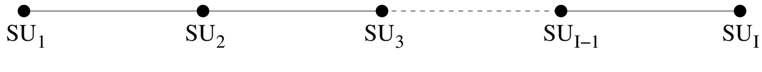


Figure 3: Chain CRN scenario: interference graph.

The socially optimal solution sees in this case users SU_1, SU_3, SU_5, ... transmit all their traffic on one channel, while users SU_2, SU_4, SU_6, ... transmit exclusively on the other, with a total cost of I. At the Nash equilibrium, instead, all users split their traffic equally on the available channels, thus leading to a cost equal to 3/2 for players SU_2, SU_3,

..., SU_{I-1}, and equal to 1 for external users SU_1 and SU_I. Therefore, the PoA has in this case the following expression:

$$PoA = \frac{3/2(I-2)+2}{I} = \frac{3I-2}{2I} \tag{33}$$

which increases with I, and is upper bounded by 3/2 for $I \to \infty$. Hence, increasing the number of secondary users leads to more inefficient network behaviors.

6.2. Non-Elastic Secondary Users: Random CRN Scenarios

Random cognitive radio network scenarios are obtained using a custom generator which considers a square area with edge equal to 1000, and randomly extracts the position of I nodes, each corresponding to a SU. A SU i interferes with SU k only if this latter is at a distance not greater than R, the interference range of i. We assume for simplicity that such range is the same for all secondary users.

For each random CRN scenario, the results are obtained averaging each point on 3000 instances, thus obtaining very small 95% confidence intervals, which are not shown in the figures for the sake of clarity.

6.2.1. Effect of the number of SUs (I)

We first evaluate the effect of the number of SUs on the Price of Anarchy in random CRN scenarios with $N = 2$ wireless channels; all SUs are characterized by the same traffic demand $r^i = 1, \forall i \in U$. We consider several R values, in the 0 to 1500 range, thus increasing the interference between SUs.

Figure 4 shows the average PoA in function of the interference range for different I values ($I \in [2, 20]$). It can be observed that for both small and high interference ranges, the PoA is very small (i.e., close to 1). These two scenarios correspond, respectively, to complete absence of interference and full interference between SUs. In the first case, obviously, the Nash equilibrium coincides with the social optimum; the same happens for the full interference case, as we have discussed before for the square interference pattern of Figure 2(a). Partial interference (i.e., intermediate R values) leads to larger gaps between Nash equilibria and optimal solutions, as it was already observed for the corresponding scenario of Figure 2(b), even though the average PoA is in all cases quite small (always less than 1.16).

Furthermore, the PoA increases when the number of secondary users becomes larger, as in the chain-like interference scenario. We can therefore argue that it is more difficult to coordinate several secondary users, thus leading to more inefficient network equilibria. However, we must further observe that the worst PoA we could measure in all the considered instances, and for all I values, was equal to 1.606, which is still an acceptable loss of efficiency with respect to the achievable network optimum.

Quadratic Cost Functions. A variation of the previous CRN scenario is considered, with quadratic cost functions ($\beta(n) = 2, \forall n \in V$); all other parameters are set as in the affine cost function scenario. Figure 5 shows the average PoA as a function of R, for I ranging

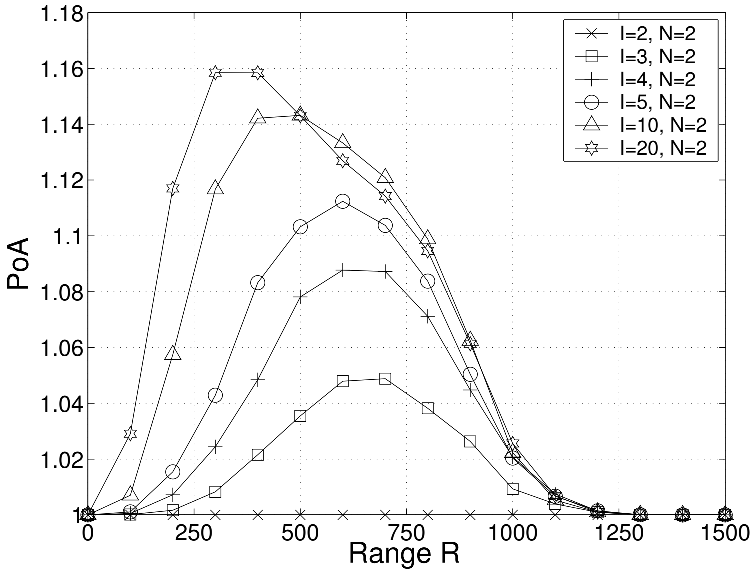


Figure 4: Non-Elastic Demands: average PoA measured in CRN scenarios with $N = 2$ available wireless channels and different numbers of SUs ($I \in [2, 20]$).

from 2 to 20. It can be observed that the PoA obtained with quadratic cost functions exhibits a trend similar to that illustrated in Figure 4 for affine cost functions. However, absolute values are higher (up to 1.4 in the Figure); as we already observed in the 4-user scenario of Section 6.1, in fact, increasing $\beta(n)$ leads to a higher Price of Anarchy.

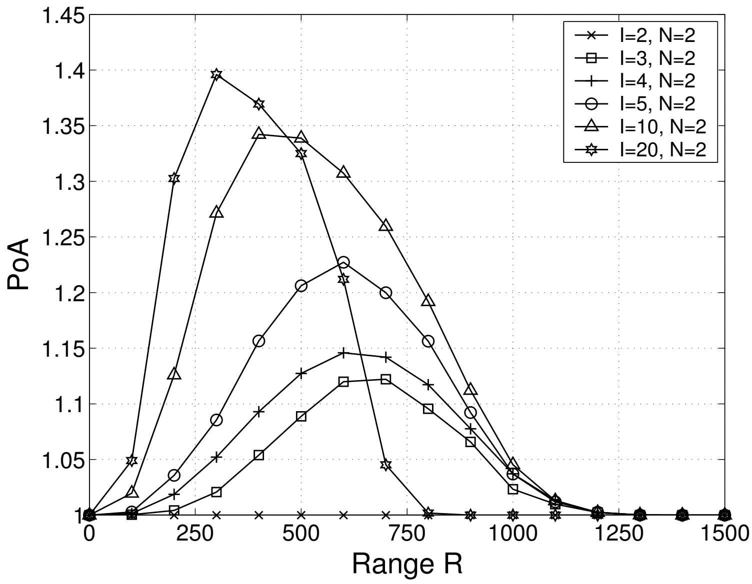
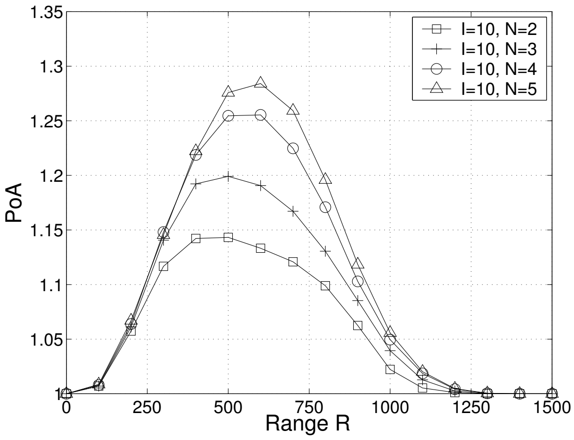


Figure 5: Non-Elastic Demands: average PoA measured in CRN scenarios with $N = 2$ available wireless channels, different numbers of SUs ($I \in [2, 20]$) and quadratic cost functions ($\beta(n) = 2$).

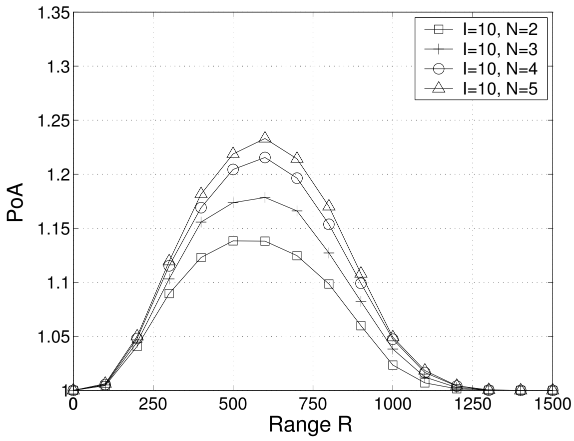
6.2.2. Effect of the number of wireless channels (N)

We then increase the number of wireless channels from 2 to 5, fixing the number of SUs to 10; the interference range R varies from 0 to 1500.

Figure 6(a) illustrates the average PoA obtained in such scenarios. It can be observed that such performance figure increases with the number of wireless channels. Intuitively, this is due to the fact that when N increases, the strategy space of the SUs also increases, thus leading to potentially worse Nash equilibria. Therefore, the availability of a larger number of wireless channels further amplifies the loss of efficiency, which can be observed especially for intermediate R values (i.e., $250 \leq R \leq 750$).



(a) $r^i = 1$



(b) $r^i = \{1/2, 3/2\}$

Figure 6: Non-Elastic Demands: average PoA measured in CRN scenarios with $I = 10$ users and different numbers of available wireless channels ($N \in [2, 5]$); (a) 10 SUs all having $r^i = 1$, and (b) 10 heterogeneous SUs (5 SUs have $r^i = 1/2$ and the others have $r^i = 3/2$).

6.2.3. Effect of the transmission rate (r^i)

We now consider a variation of the previous scenario, assuming that 5 users offer a rate equal to 1/2, while the other 5 have $r^i = 3/2$, thus maintaining the same total offered traffic (equal to 10), for the sake of comparison.

If we compare the results measured in this scenario, reported in Figure 6(b), with those obtained for homogeneous traffic demands (Figure 6(a)), we observe that the PoA is always smaller when traffic demands are different. This is essentially due to the fact that the quality of the equilibria is more influenced by the choices of “elephant” users (those who offer $r^i = 3/2$), and less by those of “mouse” users ($r^i = 1/2$). Hence, the network behavior is closer to that of a cognitive radio network with a smaller number of SUs and, as we observed before, when the number of such users decreases, the PoA also decreases (see Figure 4).

6.3. Elastic Secondary Users: 4-User CRN Scenarios

We now consider secondary users with *elastic* traffic demands. To illustrate the effect of the interference on the PoA with elastic traffic demands, we consider the same square interference patterns of Figures 2(a) and 2(b), first assuming that all SUs have affine utilities of the form $Q^i(f^i) = \sum_{n \in V} \alpha^i_n f^i_n$, where the utility value per flow unit, α^i_n, is the same for each frequency band ($\alpha^i_n = \alpha^i = 4, \forall i \in U, n \in V$). Moreover, $m^i = 0$ and M^i is sufficiently large not to limit the transmitted flow.

For the full interference scenario, we have now a $PoA = 1.562$, while for the partial interference scenario we have $PoA = 1$. Hence, differently from the non-elastic case, the PoA increases consistently when the interference between secondary users increases.

This is due to the fact that, with full interference, the solution which maximizes the social welfare is very unfair in the presence of elastic users: it allocates the available spectrum bands to a few SUs, while assigning the rest no bandwidth at all. This solution maximizes the overall users utility, but it is far from the one given by the Nash equilibrium, which distributes the network capacity more fairly among secondary users.

More specifically, in the full interference scenario, the socially optimal solution allocates 2 bandwidth units to SU_1 on both channels 1 and 2, while all other three players do not send any traffic at all. This maximizes the total utility (equal to 8), which coincides with the utility of SU_1; at the same time, such spectrum allocation is evidently unfair since all network resources are allocated to a single user. We observe that such behavior does not occur for non-elastic traffic, since in that case each user i must transmit all r^i bandwidth units.

On the other hand, at the Nash equilibrium all players send 0.8 bandwidth units on both available channels, thus leading to a perfectly fair spectrum allocation with total utility equal to 5.12. Hence, the PoA is in this case equal to $\frac{8}{5.12} = 1.56$.

The same behavior can be observed also for *logarithmic* utility functions, even if the PoA is in this case very close to 1 for all the considered parameter values. We considered several settings for the g^i parameter in the logarithmic utility function $Q^i(f^i) = g^i log(1 + h^i f^i)$, fixing $h^i = 10, \forall i \in U$, and Table 2 reports the PoA measured for the partial and full interference scenarios.

Table 2: 4-User CRN Scenarios: PoA measured for elastic users with logarithmic utility function for different values of the g^i parameter. Both full and partial interference between SUs are considered (see Figures 2(a) and 2(b), respectively)

g^i	Partial Interference	Full Interference
100	1.013	1.019
50	1.015	1.021
25	1.016	1.024

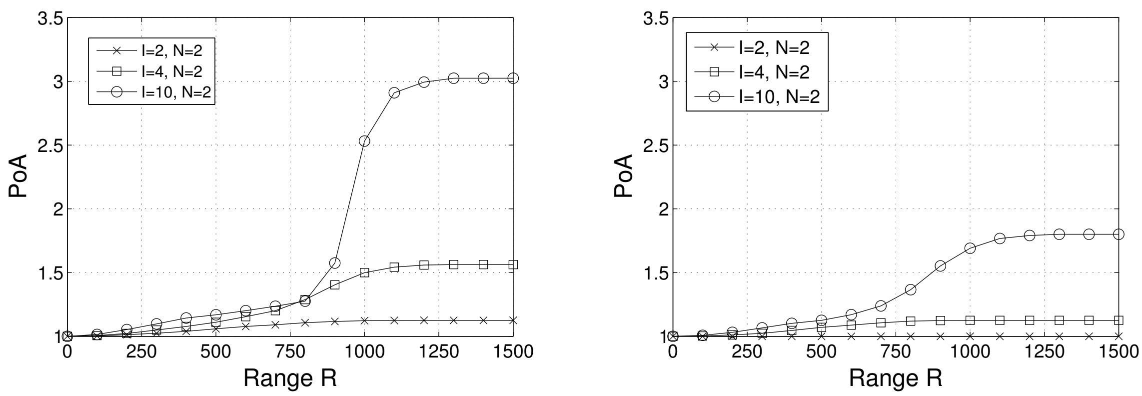
6.4. Elastic Secondary Users: Random CRN Scenarios

Random cognitive radio network scenarios are obtained by the same generator used in the case of non-elastic traffic instances. As for non-elastic scenarios, the results are obtained with very small 95% confidence intervals, which are not shown in the figures for the sake of clarity.

6.4.1. Affine Utility: Effect of the Utility value α_n^i

To gauge the impact of the utility value on the efficiency of our game, we first consider a homogeneous case in which all SUs have affine utility functions with identical utility values per unit of flow ($\alpha_n^i = 4, \forall i \in U, n \in V$); then, we analyze a CRN where 50% of the SUs have $\alpha_n^i = 2$ and the rest has $\alpha_n^i = 6$, so that the average utility value is the same as in the homogeneous case. The number of wireless channels is equal to 2, and we vary the number of SUs in the 2 to 10 range.

Figure 7 shows that the PoA of the homogeneous SUs (Figure 7(a)) is always greater than that measured in the heterogeneous case (Figure 7(b)), in line with what already observed for non-elastic scenarios (see Figure 6). Furthermore, the loss of efficiency can in this case be larger than for non-elastic traffic, especially when a large number of SUs are transmitting.



(a) $\alpha = 4$ (b) $\alpha \in \{2, 6\}$

Figure 7: Elastic demands with affine utility: average PoA measured in a CRN scenario with 2 available channels and different numbers of SUs ($I \in [2, 10]$); (a) all SUs have the same utility value ($\alpha = 4$), and (b) 50% of the SUs have $\alpha_n^i = 2$ and the rest has $\alpha_n^i = 6$.

6.4.2. Affine Utility: Effect of the Cost Parameter a_n on the individual SU utility

We now examine the behavior of the SUs under different values of a_n, the per flow unit cost parameter for wireless channel n. To this aim, we consider a random cognitive radio scenario with 3 SUs and 2 wireless channels, increasing the interference range of secondary users.

Figure 8 reports the average SUs' utility (i.e., the objective function value maximized by each elastic user) as a function of the interference range for $a_n = 1$, 2 and 4. As expected, elastic SUs send less traffic and have lower utilities as a_n increases. Furthermore, the utility decreases consistently with increasing a_n values (that is, with increasing link costs).

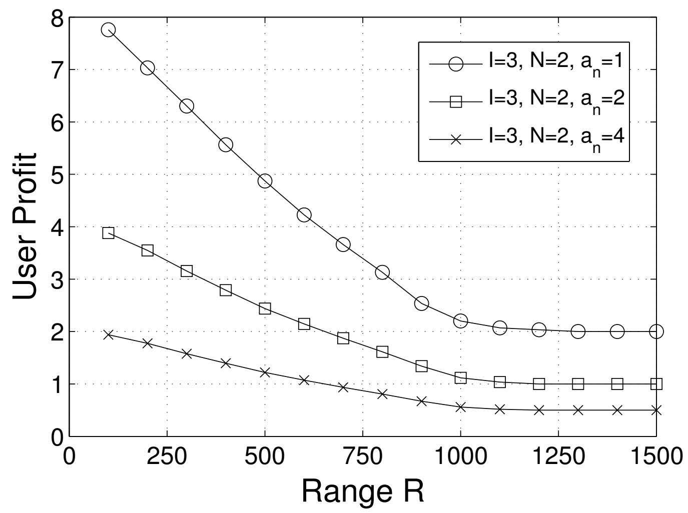


Figure 8: Elastic Demands with affine utility: average user's profit measured in a CRN scenario with 3 SUs and 2 wireless channels, with different a_n values (viz., $a_n = 1$, 2 and 4).

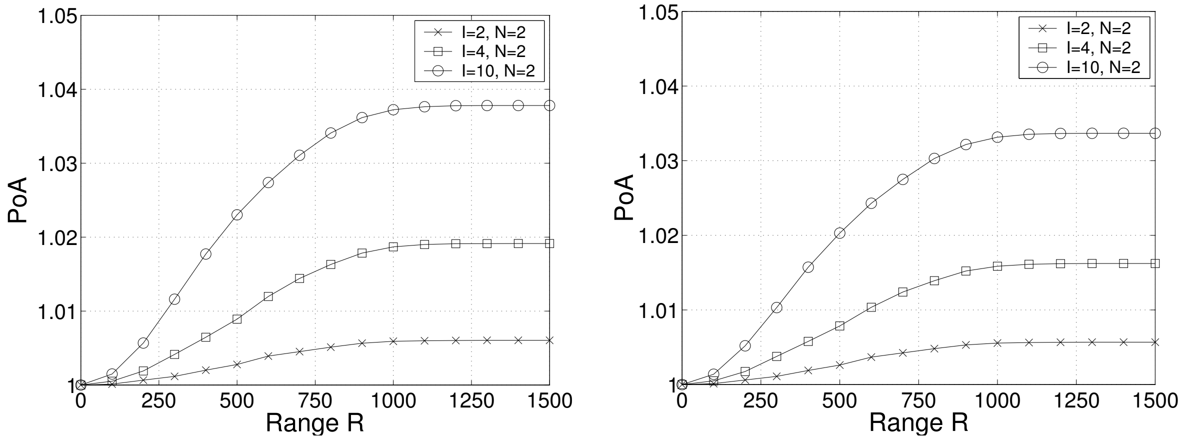
6.4.3. Logarithmic Utility: Effect of the Utility value g^i

We then consider logarithmic utility functions, starting with a homogeneous case in which all SUs have identical g^i values, and then analyzing a CRN where 50% of the SUs have $g^i = 50$ and the rest has $g^i = 150$, so that the average utility value is the same as in the homogeneous case. Figure 9 shows that also for logarithmic utility functions the PoA of the homogeneous SUs (Figure 9(a)) is always greater than that measured in the heterogeneous case (Figure 9(b)), thus confirming what already observed for affine utility functions. However, the PoA measured in this scenario is considerably smaller (always less than 1.04) than that observed for affine utility functions. Hence we can affirm that, in this case, the performance degradation due to the increasing interference level is almost negligible.

6.4.4. Logarithmic Utility: Effect of the Number of SUs (I)

We now consider the same CRN scenario of Section 6.2.1, with elastic SUs having logarithmic utility functions ($g^i = 100$ and $h^i = 10$ for all users). Figure 10 illustrates the average PoA measured for all the considered transmission ranges, for a number of users I in the 2 to 20 range.

The PoA increases with increasing I values, but remains always below 1.06, thus confirming that, in such case, the network configuration reached by SUs in a fully distributed way is very close to the optimum that could be planned in a centralized way.



(a) $g^i = 100$ (b) $g^i \in \{50, 150\}$

Figure 9: Elastic demands with logarithmic utility: average PoA measured in a CRN scenario with 2 available channels and different numbers of SUs ($I \in [2, 10]$); (a) all SUs have the same utility value ($g^i = 100$), and (b) 50% of the SUs have $g^i = 50$ and the rest has $g^i = 150$.

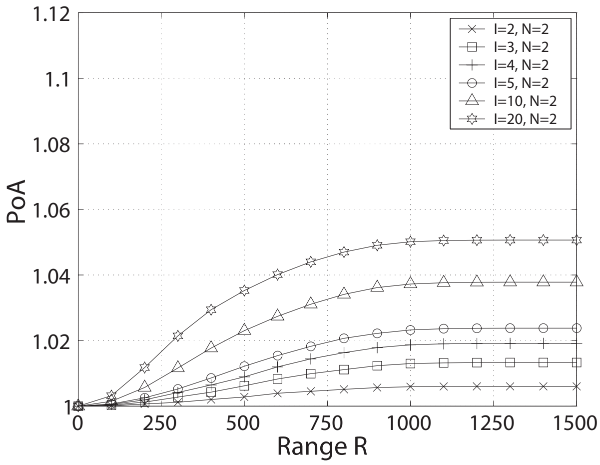


Figure 10: Elastic demands with logarithmic utility: average PoA measured in a CRN scenario with 2 available channels and different numbers of SUs ($I \in [2, 20]$).

6.4.5. Logarithmic Utility: Effect of the Number of Wireless Channels (N)

We further consider a scenario with 10 elastic SUs with logarithmic utility, having the same configuration parameters reported before, and increasing the number of available channels n from 2 to 5. Figure 11 reports the PoA measured in such case.

As already observed for non-elastic traffic, increasing the number of available channels (and consequently the strategy space) leads to higher PoA values. This degradation, however, is very limited for elastic users, and the PoA always assumes very small values, close to 1 for all the considered scenarios.

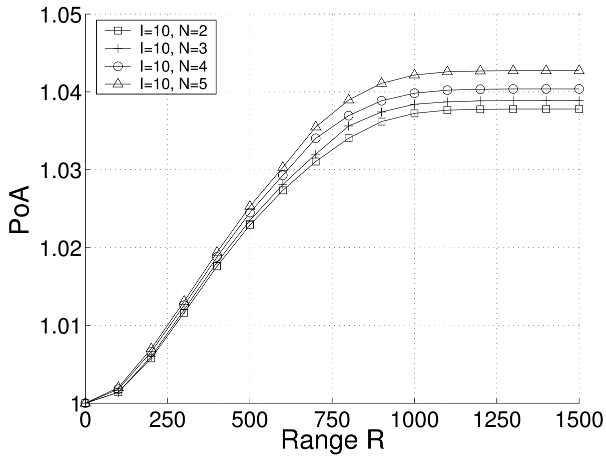


Figure 11: Elastic demands with logarithmic utility: average PoA measured in a CRN scenario with 10 SUs and different numbers of available channels ($N \in [2, 5]$).

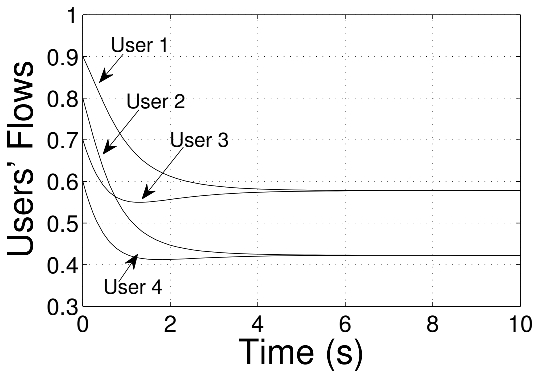
6.5. Replicator Dynamics: 4-User CRN scenarios

Finally, we investigate in this section the stability of the Nash equilibria of the spectrum access game illustrated in Section 5.4, where SUs adapt their strategies using replicator dynamics, based on the congestion level measured on each link. To this end, we consider the 4-User CRN scenarios with full interference (Figure 2(a)) and partial interference (Figure 2(b)), described in Section 6.1.1, and we assume that SUs use replicator dynamics. We recall that in such scenario, each non-elastic user i must transmit a total amount of flow $r^i = 1$ on the 2 available channels.

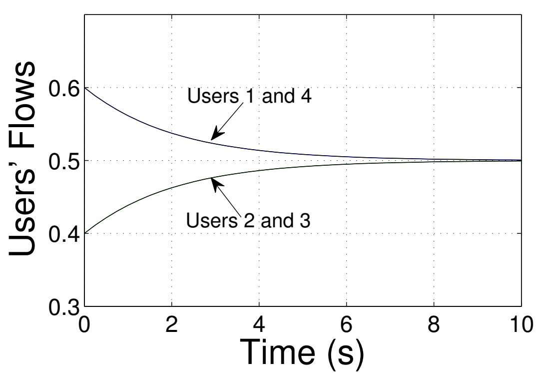
Figures 12(a) and 12(b) illustrate the convergence of SUs' flows to a Nash equilibrium, under replicator dynamics, in the case of partial and full interference between SUs, respectively. For sake of clarity, the Figures report only the amount of traffic sent by each user on wireless channel 1, since all the rest of the traffic is transmitted on channel 2.

In the partial interference scenario, we consider an example case in which, at the starting point, SU_1, SU_2, SU_3 and SU_4 send respectively 90%, 80%, 70% and 60% of their traffic on channel 1 and the rest on channel 2. It can be observed that SU_1 and SU_3 flows converge to an equilibrium point of approximately 0.58 on channel 1 and 0.42 on channel 2, while the flows of SU_2 and SU_4 converge to 0.42 and 0.58 on channels 1 and 2, respectively. Such users' behavior was already observed in Section 6.1.1, where at the equilibrium, SU_1 and SU_3 send p traffic units on channel 1 and $1 - p$ on channel 2, while SU_2 and SU_4 do the opposite, sending $1 - p$ on channel 1 and p on channel 2.

If we consider now the full interference scenario (Figure 12(b)), we observe that the SUs' flows always converge to the equilibrium point (0.5, 0.5) on channels 1 and 2, splitting equally their traffic on the 2 available channels, and this happens independently of the starting point chosen for such flows.



(a) Partial interference



(b) Full interference

Figure 12: Non-Elastic Demands: Convergence of SUs' flows (transmitted on wireless channel 1) to a Nash equilibrium point under replicator dynamics in the 4-User CRN scenarios with (a) partial interference and (b) full interference between secondary users.

7. Conclusion

This paper addressed the spectrum access problem in cognitive radio networks from a game theoretical perspective. The problem has been modeled as a non-cooperative game where secondary users access simultaneously multiple spectrum bands left available by primary users, optimizing their objective function which takes into account the congestion-dependent cost functions.

As a key innovative feature with respect to existing works, we modeled accurately the interference between SUs, capturing the effect of spatial reuse, and we also used effective congestion cost functions that ensure good Nash equilibria. We considered both elastic and non-elastic traffic demands, modeling data transfers and real-time applications. Furthermore, we demonstrated the existence of the Nash equilibrium, and we computed equilibrium flow settings. We considered an alternative formulation of the dynamic spectrum access problem based on population games and replicator dynamics. Finally, we performed a thorough numerical analysis of the proposed model, studying the impact of several parameters, like the number of SUs and wireless channels as well as the interference between SUs, on the game efficiency. Our results indicate that:

- the cost functions adopted in this paper enable good Nash equilibria, thus representing a good starting point for designing pricing mechanisms that foster cooperation in cognitive radio networks.
- The PoA depends significantly on the interference between SUs and increases with both the number of SUs and that of wireless channels.
- For non-elastic traffic, the PoA is higher with partial interference between users than with full interference. An opposite result holds for elastic traffic since, in this

case, the solution which maximizes the social welfare is generally quite unfair, and hence far from the Nash Equilibrium solutions that provide more fair spectrum allocations.

- Under replicator dynamics, non-elastic SUs' flows converge always to the Nash equilibrium points of the spectrum access game.

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Appendix A. Proof of Proposition 1

In this appendix, we demonstrate that the spectrum access game proposed in Section 5.4 (with $\beta(n) = 1$, $f_n^{PU} = 0$, $\forall n \in V$, and $a_{k,i}^n = a_{i,k}^n$, $\forall i, k \in U, n \in V$, i.e., the interference matrix is symmetric), belongs to the class of *potential games* [25], and admits the following potential function Φ:

$$\Phi = \frac{1}{2}\sum_{i\in U} OF_N^i + \frac{1}{2}\big(\sum_{i\in U}\sum_{n\in V}[a_n(f_n^i)^2 + b_n f_n^i]\big). \tag{A.1}$$

We recall that the function Φ is a *potential function* if it satisfies (for each player i, each multi-strategy z and each strategy f^i) the following condition:

$$\Phi(f^i, f^{-i}) - \Phi(z^i, f^{-i}) = OF_N^i(f^i, f^{-i}) - OF_N^i(z^i, f^{-i}). \tag{A.2}$$

Let $\Phi_n(f_n)$ and $OF_{Nn}^i(f_n)$ be the potential function and the user cost function, respectively, per wireless channel n. Hence, $\Phi = \sum_{n\in V}\Phi_n(f_n)$ and $OF_N^i = \sum_{n\in V} OF_{Nn}^i(f_n)$. The function Φ_n has the following expression:

$$\begin{aligned}\Phi_n(f_n^i, f_n^{-i}) =& \frac{1}{2}\big(\sum_{i\in U}\sum_{k\in U} a_n a_{k,i}^n f_n^i f_n^k + \sum_{i\in U}[a_n(f_n^i)^2 + b_n f_n^i]\big) \\ =& \frac{a_n}{2}\sum_{j\in U, j\neq i}\sum_{k\in U, k\neq i} a_{k,j}^n f_n^j f_n^k + \frac{a_n}{2}\sum_{k\in U, k\neq i} a_{k,i}^n f_n^k f_n^i + \frac{a_n}{2}\sum_{K\in U, k\neq i} a_{i,k}^n f_n^k f_n^i + \\ & \frac{a_n}{2} a_{i,i}^n f_n^i f_n^i + \frac{a_n}{2}\sum_{k\in U, k\neq i}(f_n^k)^2 + \frac{a_n}{2}(f_n^i)^2 + \frac{b_n}{2}\sum_{k\in U, k\neq i} f_n^k + \frac{b_n}{2} f_n^i.\end{aligned} \tag{A.3}$$

Recall that $a^n_{k,i} = a^n_{i,k}, \forall i, k \in U, n \in V$, and $a^n_{i,i} = 1, \forall i \in U, n \in V$ (since each SU i obviously "interferes" with himself on each wireless channel n).

If we focus on a generic channel n, we can verify that any unilateral deviation of SU i on such channel is exactly equal to the difference in function $\Phi_n(f_n)$. In fact, the following equality holds:

$$\begin{aligned}\Phi_n(f^i_n, f^{-i}_n) - \Phi_n(z^i_n, f^{-i}_n) =& a_n \sum_{k \in U, k \neq i} a^n_{k,i} f^k_n (f^i_n - z^i_n) + a_n((f^i_n)^2 - (z^i_n)^2) + b_n(f^i_n - z^i_n) \\ =& a_n((f^i_n)^2 - (z^i_n)^2) + (a_n \sum_{k \in U, k \neq i} a^n_{k,i} f^k_n + b_n)(f^i_n - z^i_n) \\ =& OF^i_{Nn}(f^i_n, f^{-i}_n) - OF^i_{Nn}(z^i_n, f^{-i}_n).\end{aligned} \tag{A.4}$$

Then, by summing up over all wireless channels n, we prove that Equation (A.2) holds, and therefore that the spectrum access game admits Φ as a potential function.