Three-valued possibilistic networks:
Semantics & inference

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***Abstract*—Possibilistic networks are belief graphical models based on possibility theory. This paper deals with a special kind of possibilistic networks called three-valued possibilistic networks where only three possibility levels are used to encode uncertain information. The paper analyzes different semantics of three-valued networks and provides precise relationships relating the different semantics. More precisely, the paper analyzes two categories of methods for deriving a three-valued joint possibility distribution from a three-valued possibilistic network. The first category of methods is based on viewing the three-valued possibilistic network as a family of compatible networks and defining combination rules for deriving the three-valued joint distribution. The second category is based on three-valued chain rules using three-valued operators inspired from some three-valued logics. Finally, the paper shows that the inference using the well-known Junction tree algorithm cannot be extended to all the three-valued chain rules.**

***Keywords*-Possibilistic networks, incomplete information;**

I. INTRODUCTION

Possibility theory is a well-known framework for dealing with uncertain and incomplete information. Possibilistic networks [1] are powerful tools for compactly encoding joint possibility distributions. They can be viewed as counterparts of Bayesian networks using possibility theory. Several alternative frameworks are proposed to encode ill-known beliefs like lower and upper probability expectations, belief functions and evidence theory [2], fuzzy sets, imprecise probabilities [3], etc. Interval-based representations are widely adopted [4] but their major problem is their high computational complexity for inferring posterior probability intervals while in practice the obtained intervals are often too large to be exploited [5]. Min-based possibilistic networks can show different behaviors due to the idempotence property of the $\min$ and $\max$ operators.

In [6], we proposed three-valued possibilistic networks where the only used possibility degrees are 0, 1 and another value to encode uncertain beliefs. In the literature, especially in three-valued logics, a third "truth" value is used to encode some special kinds of ignorance or ill-known beliefs. In [7], the author addresses some issues and shows the limits of the use of some three-valued logics for dealing with epistemic uncertainty. Three-valued possibilistic networks encode uncertain information and they are viewed in [6] as families of compatible possibilistic networks while the extended three-valued chain rule uses only Kleene's conjunction operator.

In this paper, we go one step further. We provide semantics analysis of three-valued possibilistic networks encoding imprecise and ill-known beliefs. We provide two categories of semantics: the first one is based on families of compatible networks while the second one is based on extending the chain rule to the three-valued setting. We analyze several three-valued conjunction operators from well-known three-valued logics and provide precise relationships between the different semantics. Finally, the paper shows that inference based on the well-known junction tree algorithm [13] cannot be extended to all the three-valued chain rules. Before presenting these results, let us provide brief refresher on possibilistic networks.

II. BRIEF REFRESHER ON POSSIBILITY THEORY AND POSSIBILISTIC NETWORKS

A. Possibility theory

Possibility theory [9][10] is an alternative uncertainty theory suitable for dealing with uncertain and incomplete knowledge. This framework uses two dual measures (possibility and necessity) to assess the knowledge/ignorance. One of the fundamental concepts of possibility theory is the one of possibility distribution π which is a mapping from the universe of discourse Ω to the interval $[0, 1]$. A possibility degree $\pi(w_i)$ expresses to what extent an elementary event $\omega_i \in \Omega$ can be the actual state of the world. Hence, $\pi(w_i)$=1 means that w_i is totally possible and $\pi(w_i)$=0 denotes an impossible event. The relation $\pi(w_i) > \pi(w_j)$ means that w_i is more possible than w_j. A possibility distribution π is normalized if $\max_{w_i \in \Omega} \pi(w_i)$=1.

A possibility measure $\Pi(\phi)$ evaluates the possibility degree relative to an event $\phi \subseteq \Omega$. It is defined as follows:

$$\Pi(\phi) = \max_{w_i \in \phi}(\pi(w_i)). \quad (1)$$

The necessity measure evaluates the certainty entailed by the current knowledge of the world encoded by the possibility distribution π:

$$N(\phi) = 1 - \Pi(\overline{\phi}) = 1 - \max_{w_i \notin \phi}(\pi(w_i)), \quad (2)$$

where $\overline{\phi}$ denotes the complementary of ϕ in Ω. In possibility theory there are several interpretations for the scale [0,1]. Accordingly, there are two variants of possibility theory:

- Qualitative (or min-based) possibility theory where the possibility measure is a mapping from the universe of discourse Ω to an "ordinal" scale where only the "ordering" of values is important.

- Quantitative (or product-based) possibility theory: Here, the possibilistic scale [0,1] is numerical and possibility degrees are like numeric values that can be manipulated by arithmetic or comparative operators depending on the interpretation of the unit scale.

This work focuses only on the qualitative setting.

B. Possibilistic networks

A possibilistic network ΠG=<G,Θ> is specified by:

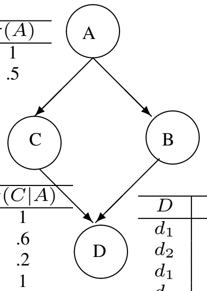
i) *A graphical component* G consisting in a directed acyclic graph (DAG) where vertices represent variables of interest and edges represent direct *dependence* relationships between these variables. Each variable A_i is associated with a domain D_i containing the values a_i that can be taken by the variable A_i.

ii) *A quantitative component* Θ allowing to quantify the uncertainty relative to the relationships between domain variables using local possibility tables (CPTs). The possibilistic component or ΠG's parameters consist in a set of local possibility tables Θ_i=$\{\theta_{a_i|u_i}\}$ where $a_i \in D_i$ and u_i is an instance of U_i denoting the parent variables of A_i in the network ΠG.

Note that all the local possibility distributions Θ_i must be normalized, namely $\forall i$=1..n, $\forall u_i \in D_{U_i}$, $\max_{a_i \in D_i}(\theta_{a_i|u_i})$=1.

Example 1

Figure 1 gives an example of a possibilistic network over four binary variables A, B, C and D.



A	$\pi(A)$
a_1	1
a_2	.5

B	A	$\pi(B\|A)$
b_1	a_1	1
b_2	a_1	1
b_1	a_2	.2
b_2	a_2	1

C	A	$\pi(C\|A)$
c_1	a_1	1
c_2	a_1	.6
c_1	a_2	.2
c_2	a_2	1

D	B	C	$\pi(D\|BC)$
d_1	b_1	c_1	0
d_2	b_1	c_1	1
d_1	b_1	c_2	1
d_2	b_1	c_2	.4
d_1	b_2	c_1	.6
d_2	b_2	c_1	1
d_1	b_2	c_2	1
d_2	b_2	c_2	.4

Figure 1. Example of a possibilistic network.

In the min-based possibilistic setting, the joint possibility distribution is factorized using the min-based chain rule defined as follows:

$$\pi(a_1, a_2, .., a_n) = \min_{i=1}^{n}(\pi(a_i|u_i)). \quad (3)$$

Example 1 (continued)

For the network of Figure 1, the joint possibility distribution is derived using the following formula:

$$\pi(A, B, C, D) = \min(\pi(A), \pi(C|A), \pi(B|A), \pi(D|BC)). \quad (4)$$

III. THREE-VALUED POSSIBILISTIC NETWORKS: BASIC DEFINITIONS

In this section, we provide some definitions needed for understanding the rest of the paper.

Definition 1: B-possibility distribution
A B-possibility distribution over a universe of discourse Ω is a boolean possibility distribution where $\forall \omega \in \Omega$, $\pi_B(\omega) \in \{0, 1\}$.

A boolean possibility distribution allows to encode only fully accepted (completely possible) interpretations or fully rejected (completely impossible) ones.

Definition 2: C-possibility distribution
A C-possibility distribution over a universe of discourse Ω is a normalized possibility distribution where $\forall \omega \in \Omega$, $\pi_C(\omega) \in \{0, 1, C\}$ where C encodes conflicting or imprecise beliefs.

In Definition 2, an interpretation ω can be associated with a possibility degree of 0 (namely ω is excluded), 1 (namely ω is fully accepted) or C meaning that the corresponding information is imprecise or conflicting. For instance, in case of multi-source information, one source may associate 0 with ω while another source may associate 1 with the same interpretation ω. The underlying semantic is that we have information but it is either imprecise or conflicting. Note that the value C is not an intermediary value between 0 and 1. Let us now define the concept of compatible distribution.

Definition 3: Let π_B be a boolean possibility distribution over Ω. π_B is compatible with a C-possibility distribution π_C iff:

Condition 1: $\forall \omega \in \Omega$, $\pi_B(\omega) =^c \pi_C(\omega)$.

Condition 2: $\max_{\omega \in \Omega}(\pi_B(\omega))$=1.

Condition 1 in Definition 3 ensures that the possibility degree of any interpretation ω is among the ones allowed by the C-possibility distribution π_C. Namely, $\pi_B(\omega) =^c \pi_C(\omega)$ means $\pi_B(\omega)$=1 if $\pi_C(\omega)$=1, $\pi_B(\omega)$=0 if $\pi_C(\omega)$=0 and $\pi_B(\omega)$=1 or $\pi_B(\omega)$=0 if $\pi_C(\omega)$=C. Condition 2 ensures that the compatible distribution π_B is normalized.

As for a C-possibility distribution π_C, it is said normalized if it has at least one boolean distribution π_B that is compatible with π_C. In this paper, we consider only normalized distributions.

Example 2

Let π_C be a C-possibility distribution on $\Omega=\{\omega_1, \omega_2, \omega_3\}$.

Ω	$\pi_C(\omega)$
ω_1	1
ω_2	0
ω_3	C

Ω	$\pi'_B(\omega)$
ω_1	1
ω_2	0
ω_3	1

Ω	$\pi''_B(\omega)$
ω_1	1
ω_2	0
ω_3	0

Table I
EXAMPLE OF A C-POSSIBILITY DISTRIBUTION π_C AND TWO COMPATIBLE DISTRIBUTIONS π'_B AND π''_B.

It is clear that with a C-possibility distribution containing n entries having the value C, there are at most 2^n compatible boolean distributions. In the following, we define possibilistic networks allowing to encode uncertainty and imprecise/conflicting beliefs.

IV. C-POSSIBILISTIC NETWORKS: SYNTAX AND SEMANTICS

Three-valued possibilistic networks for encoding imprecise beliefs are originally proposed in [6]. In this paper, we call such networks C-possibilistic networks[1] which are graphical belief models allowing a compact encoding of imprecise joint possibility distributions. This is in the spirit of multiple source possibilistic logic [11] where the uncertainty associated with each formula is composed of all the uncertainty degrees of the sources regarding this formula.

A. *C-possibilistic networks*

In C-possibilistic networks (C-PNs for short), the uncertainty levels that are allowed are 0 (expressing the fact that the agent believes that the event is impossible), 1 (to encode totally possible events) and C to encode for instance conflicting information. In [6], the C value is encoded by $\{0, 1\}$. Recall that this value encodes the fact that the belief degree is imprecise in the sense that it is either 0 or 1 but there is no way to know it.

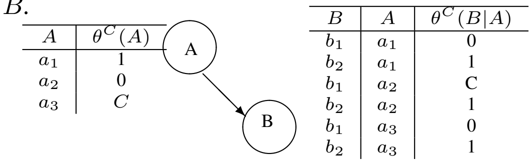
Definition 4: A three-valued C-possibilistic network G^C=<G,Θ^C> is a graphical model such that

1) G=<V, E> is a directed acyclic graph (DAG) over the set of variables $V=\{A_1, .., A_n\}$ and E denotes edges between variables of V.
2) $\Theta^C=\{\theta^C_1, .., \theta^C_n\}$ a set of local C-possibility tables where each θ^C_i denotes a local three-valued possibility distribution associated with the variable A_i in the context of its parents U_i. For each configuration u_i of U_i the parents of variable A_i, there exists at least $a_i \in D_{A_i}$ such that $\theta^C_i(a_i|u_i)$=1 or $\theta^C_i(a_i|u_i)$=C.

In Definition 4, the plausibility of any A_i's value in the context of any of its parents' configuration, denoted $\theta^C_{a_i|u_i}$, can be either 0, 1 or C. If the network contains only values 0 and 1, then such a network is called a boolean network.

Example 3

Figure 2 gives an example of a C-PN over two variables A and B.



A	$\theta^C(A)$
a_1	1
a_2	0
a_3	C

B	A	$\theta^C(B\|A)$
b_1	a_1	0
b_2	a_1	1
b_1	a_2	C
b_2	a_2	1
b_1	a_3	0
b_2	a_3	1

Figure 2. Example of a C-PN.

In the following sections, we address two semantics associated with C-PNs. The first semantics is to view a C-PN as a set of compatible boolean possibilistic networks. The second semantics is based on extending the min-based chain rule of Equation 3 to the three-valued setting.

B. *C-PNs as families of compatible boolean possibilistic networks*

A boolean possibilistic network G^B is compatible with the C-PN G^C according to the following definition.

Definition 5: Let G^C=<G,Θ^C> be a C-PN. A boolean network G^B=<G,Θ^B> is compatible with G^C iff

1) G^C and G^B have exactly the same graph and
2) $\forall \theta^B_{a_i|u_i} \in \Theta^B$, $\theta^B_{a_i|u_i} =^c \theta^C_{a_i|u_i}$ with $\theta^C_{a_i|u_i} \in \Theta^C$.

According to Definition 5, a possibilistic network G^B is compatible with a C-PN G^C if they have the same structure and every local possibility distribution $\theta^B_{a_i|u_i}$ of G^B is compatible with its corresponding local C-distribution $\theta^C_{a_i|u_i}$ in G^C.

Let us now consider how to compute the uncertainty degree of any event $\phi \subseteq \Omega$ from a C-PN. Namely, how to derive a three-valued joint possibility distribution from a C-PN? Recall that a C-PN is viewed as a collection of boolean possibilistic networks G_i that are compatible with C-PN. Hence, a joint C-based possibility distribution can be computed from these compatible networks as follows:

Definition 6: Let C-PN be a C-possibilistic network and let $\mathbb{F}_{C-PN}=\{G_1, G_2, .., G_m\}$ be the set of compatible possibilistic networks with C-PN. Then $\forall \omega \in \Omega$,

$$\pi^C(\omega) = \begin{cases} \pi^{G_i}(\omega) & \text{if } \forall\ G_i, G_j \in \mathbb{F}_{C-PN} \text{ with } i \neq j,\ \pi^{G_i}(\omega) = \pi^{G_j}(\omega); \\ \text{C} & \text{otherwise.} \end{cases} \tag{5}$$

where G_i and G_j are boolean possibilistic networks compatible with the C-PN. Note that the set of compatible networks is finite.

Equation 5 is based on a kind of a unanimity rule. Namely, in case where all the compatible networks agree regarding the possibility degree associated with a given interpretation ω then π^C associates the same degree to ω. In case of disagreement, it is the value C that is associated with ω

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[1] In this paper, we prefer to use the value C to denote the set $\{0, 1\}$ only to simplify the notations.

in the joint distribution π^C.

The C-joint distribution induced using Equation 5 is normalized as stated by the following proposition:

Proposition 1: Let G^C=<G,Θ^C> be a C-PN. Let π^C the C-joint possibility distribution obtained from G^C using Equation 5. Let also $\mathbb{F}_{\pi^C}$ denote the set of boolean joint distributions that are compatible with π^C. Then,

$$\mathbb{F}_{\pi^C}\neq\emptyset.$$

Proof. It is not hard to find a normalized distribution π^B that is compatible with π^C provided that π^C is normalized. One among such distributions is the one encoded by the compatible possibilistic network where all the C values in the C-PN G^C are replaced by 1. □

C. From C-PNs to C-joint distributions using three-valued chain rules

Another way to derive the C-joint distribution associated with a C-PN consists in extending the min-based chain rule of Equation 3 to our three-valued possibilistic setting. In the following, we study if the conjunctive operators used in some three-valued logics can suit our semantics of C-PNs given in Definition 4.

Three-valued logics generally involve one truth value encoding $true$, a second value encoding $false$ and a third value encoding *undefined, unknown, undecided, maybe, both states, paradoxical* or *irrelevant state*. These systems extend boolean logic to represent and reason with some kinds of incomplete information. The first works on three-valued logics are due to Łukasiewicz in the 1920's. In addition to the values representation, the main differences between the existing three-valued logics concern the interpretation associated with the third value and the underlying connectives. In [7], the author points out the limits of some three-valued logics for dealing with beliefs and incomplete knowledge. The author interprets the truth values as epistemic values where the value $true$ encodes *the agent believes the proposition* p, the value $false$ encodes *the agent believes* $\neg p$ while the third value encodes *the agent is ignorant about* p. Note also that most of three-valued logics assume an ordering on the truth values (the third truth value U is often considered as intermediate between 0 and 1, namely 0$\leq U\leq$1).

1) Three -valued chain rule based on Kleene's conjunction: : In Kleene's three-valued logic (3K) [8], the used values are {0, $\frac{1}{2}$, 1} where the value C denotes the third truth value. To be coherent with our notations, we will use the value C instead of $\frac{1}{2}$. The conjunction and disjunction operators are defined as follows:

		A		
		0	1	C
	0	0	0	0
B	1	0	1	C
	C	0	C	C

A$\wedge^K$B

		A		
		0	1	C
	0	0	1	C
B	1	1	1	1
	C	C	1	C

A$\vee^K$B

Note that in (3K), the interpretation of the C state is an imprecise state. It is seen as a sealed box containing either the value 0 or the value 1 but there is no means to know it with certainty. Hence, the state C corresponds to the value C in C-PNs. Note that the truth tables of Łukasiewicz conjunction and disjunction and those of Gödel's ones are equivalent to Kleene's ones.

Now using the $\wedge^K$ operator, the min-based chain rule of Equation 3 is extended to the $3V$-based setting as follows: Let ω=$a_1a_2..a_n$, then

$$\pi^C_K(a_1,a_2,..,a_n)=\overset{K}{\min_{i=1..n}}(\theta^C_i(a_i|u_i)). \qquad (6)$$

Example 3 (continued)

Let us compute the C-joint distribution encoded by the network of Figure 2 using the $\wedge^K$ *min-based chain rule of Equation 6.*

A	B	$\pi^C_K(AB)$
a_1	b_1	0
a_2	b_1	0
a_3	b_1	0
a_1	b_2	1
a_2	b_2	0
a_3	b_2	C

Table II

C-JOINT DISTRIBUTION ENCODED BY THE NETWORK OF FIGURE 2 USING EQUATION 6.

It is easy to show that there are two boolean distributions which are compatible with the C-joint distribution of Table II. These two distributions are exactly those boolean distributions encoded by the compatible networks with the C-PN of Figure 2. Note that several boolean networks can encode the same joint distribution.

The C-joint distribution obtained with the three-valued chain rule based on Kleene's conjunction (using Equation 6) is equivalent to the one obtained by Equation 5 as stated by the following proposition:

Proposition 2: Let G^C=<G,Θ^C> be a C-PN. Let π^C be the C-joint possibility distribution obtained from G^C using Equation 5 and let π^C_K the C-joint possibility distribution obtained from G^C using Equation 6. Then, $\forall\omega$=$a_1a_2..a_n$,

$$\pi^C(a_1a_2..a_n)=\pi^C_K(a_1a_2..a_n).$$

Proof. Let ω=$a_1a_2..a_n$ be an interpretation (an instantiation of the network variables A_1, A_2.. A_n). Three cases can occur:

- *Case 1*: $\pi^C_K(a_1a_2..a_n)$=0
  In this case, according to the Kleene's conjunction-based chain rule of Equation 6, $\pi^C_K(a_1a_2..a_n)$=0 means that $\exists\theta^C_i\in\Theta^C$, $\theta^C_i(a_i|u_i)$=0. As a consequence, every compatible network G_j with G^C will have for the variable A_i, $\theta^C_i(a_i|u_i)$=0. Hence, $\forall G_j\in\mathbb{F}_{G^C}$, then $\pi_{G_j}(a_1a_2..a_n)$=0.

- *Case 2*: $\pi^C_K(a_1a_2..a_n)$=1
  According to the chain rule of Equation 6, $\pi^C_K(a_1a_2..a_n)$=1 means that $\forall\theta^C_i\in\Theta^C$, $\theta^C_i(a_i|u_i)$=1. So in every compatible network G_j with G^C, we have for the variable A_i, $\theta^C_i(a_i|u_i)$=1. Hence, $\forall G_j\in\mathbb{F}_{G^C}$, then $\pi_{G_j}(a_1a_2..a_n)$=1.
- *Case 3*: $\pi^C_K(a_1a_2..a_n)$=C
  This case is found only if there exists at least a local three-valued distribution $\theta^C_i\in\Theta^C$ such that $\theta^C_i(a_i|u_i)$=C. Assume that θ^C_i is the only local distribution such that $\theta^C_i(a_i|u_i)$=C. According to the definition of compatible networks, there will be only two compatible networks with G^C which are G_1 having $\theta^C_1(a_i|u_i)$=0 and G_2 having $\theta^C_2(a_i|u_i)$=0. Hence $\pi^{G_1}(a_1a_2..a_n)$=0 while $\pi^{G_2}(a_1a_2..a_n)$=1. Consequently, $\pi^C(a_1a_2..a_n)$=C.

Given that only these three cases can occur, then we can assert that for every ω=$a_1a_2..a_n$, its possibility degree $\pi^C(a_1a_2..a_n)$ computed using the combination rule of Equation 5 is equivalent to $\pi^C_K(a_1a_2..a_n)$ computed using the three-valued chain rule of Equation 6. □
Another important result is given in Proposition 5:

Proposition 3: Let $\mathbb{F}_{G^C}$ denote the set of joint distributions induced by the boolean networks that are compatible with the C-PN G^C. Let also $\mathbb{F}_{\pi^C_K}$ denote the set of boolean joint distributions π^B_i that are compatible with the C-joint distribution π^C_K obtained using the $3V$-based chain rule of Equation 6. Then,

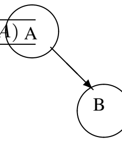
$$\mathbb{F}_{G^C} \subseteq \mathbb{F}_{\pi^C_K}.$$

It is easy to show that any distribution π in $\mathbb{F}_{G^C}$ is also in $\mathbb{F}_{\pi^C_K}$. The converse is false as it is shown in the following counter-example.

Counter-example 1

Figure 3 gives an example of a C-PN over two binary variables A and B with its C-joint possibility distribution π_K and a compatible joint distribution π^B.

A	$\theta^C(A)$
a_1	C
a_2	1



B	A	$\theta^C(B\|A)$
b_1	a_1	1
b_2	a_1	0
b_1	a_2	C
b_2	a_2	C

B	A	$\pi^C_K(AB)$
b_1	a_1	C
b_2	a_1	0
b_1	a_2	C
b_2	a_2	C

Figure 3. Example of a C-PN with its C-joint possibility distribution π_K.

Now, consider the distribution π_B of Table III.

B	A	$\pi^B(AB)$
b_1	a_1	1
b_2	a_1	0
b_1	a_2	0
b_2	a_2	0

Table III
B-JOINT DISTRIBUTION COMPATIBLE WITH THE C-JOINT DISTRIBUTION ENCODED BY THE NETWORK OF FIGURE 3.

It is clear that π^B is compatible with C-joint distribution encoded by the network of Figure 3 but there is no compatible network with the three-valued network of Figure 3 that encodes π^B.

2) Three-valued chain rule based on Bochvar's conjunction: : Bochvar's three-valued logic (3B) [12] interprets the third truth value as "meaningless", "paradoxical" or "irrelevant". This logic defines internal and external connectives as follows:

A

		0	1	C
	0	0	0	C
B	1	0	1	C
	C	C	C	C

A$\wedge^{B_I}$B

A

		0	1	C
	0	0	1	C
B	1	1	1	C
	C	C	C	C

A$\vee^{B_I}$B

A

		0	1	C
	0	0	0	0
B	1	0	1	0
	C	0	0	0

A$\wedge^{B_E}$B

A

		0	1	C
	0	0	1	0
B	1	1	1	1
	C	0	1	0

A$\vee^{B_E}$B

In the above truth tables, $\wedge^{B_I}$ (resp. $\wedge^{B_E}$) denotes internal (resp. external) conjunction and $\vee^{B_I}$ (resp. $\vee^{B_E}$) denotes internal (resp. external) disjunction. Using the $\wedge^{B_I}$ min-based operator, we define the $\wedge^{B_I}$-based chain rule as follow: Let ω=$a_1a_2..a_n$, then

$$\pi^C_{B_I}(a_1, a_2, .., a_n) = \overset{B_I}{\min_{i=1..n}}(\theta^C_i(a_i|u_i)). \qquad (7)$$

Example 3 (continued)

Let us now compute the C-joint distribution encoded by the network of Figure 2 using the min-based chain rule of Equation 7.

A	B	$\pi^C_{B_I}(AB)$
a_1	b_1	0
a_2	b_1	C
a_3	b_1	C
a_1	b_2	1
a_2	b_2	0
a_3	b_2	C

Table IV
C-JOINT DISTRIBUTION ENCODED BY THE NETWORK OF FIGURE 2 USING EQUATION 7.

The following proposition relates the compatible distributions obtained using Kleene's conjunction-based chain rule and the one based on Bochvar's internal conjunction:

Proposition 4: Let $\mathbb{F}_{\pi^C_K}$ denote the set of boolean joint distributions π^B_i that are compatible with the C-joint distribution π^C_K obtained using the $3V$-based chain rule of Equation 6. Let also $\mathbb{F}_{\pi^C_{B_I}}$ denote the set of boolean joint distributions π that are compatible with the C-joint distribution $\pi^C_{B_I}$ obtained using the $3V$-based chain rule of Equation 7. Then,

$$\mathbb{F}_{\pi^C_K} \subseteq \mathbb{F}_{\pi^C_{B_I}}.$$

Proof. Let us first show that every compatible distribution π^C_K belonging to $\mathbb{F}_{\pi^C_K}$ is necessarily in $\mathbb{F}_{\pi^C_{B_I}}$ and provide a counter example to show that the converse is false.

- Let us show that if $\pi^C_K \in \mathbb{F}_{\pi^C_K}$ then $\pi^C_K \in \mathbb{F}_{\pi^C_{B_I}}$: Let ω=$a_1a_2..a_n$ be an interpretation. In order to show that $\pi^C_K \in \mathbb{F}_{\pi^C_K}$ then $\pi^C_K \in \mathbb{F}_{\pi^C_{B_I}}$, the following cases are to be considered:
    1) If $\pi^C_K(a_1a_2..a_n)$=0 then according the definition of compatible distributions $\pi_K(a_1a_2..a_n)$ is either 0 or C. Hence, $\exists \theta^C_i \in \Theta^C$ such that $\theta^C_i(a_i|u_i)$=0 or $\theta^C_i(a_i|u_i)$=C. As a consequence, $\pi^C_{B_I}(a_1a_2..a_n)$=0 or $\pi^C_{B_I}(a_1a_2..a_n)$=C. In both cases, $\pi^C_K(a_1a_2..a_n) \underset{c}{=} \pi^C_{B_I}(a_1a_2..a_n)$.
    2) Otherwise $\pi^C_K(a_1a_2..a_n)$=1 then according the definition of compatible distributions $\pi_K(a_1a_2..a_n)$=1 or $\pi_K(a_1a_2..a_n)$=C. This mean that $\forall \theta^C_i \in \Theta^C$, $\theta^C_i(a_i|u_i)$=1 or $\theta^C_i(a_i|u_i)$=C. Then $\pi^C_{B_I}(a_1a_2..a_n)$=1 or $\pi^C_{B_I}(a_1a_2..a_n)$=C.

  We conclude that if $\pi^C_K \in \mathbb{F}_{\pi^C_K}$ then $\pi^C_K \in \mathbb{F}_{\pi^C_{B_I}}$.

- Now, let us show that $\exists \pi^C_K \in \mathbb{F}_{\pi^C_{B_I}}$ and $\pi^C_K \notin \mathbb{F}_{\pi^C_K}$: It is easy from Example of Table IV to find a compatible distribution with $\pi^C_{B_I}$ that is not included in $\mathbb{F}_{\pi^C_K}$. For instance, the distribution of Table V which is compatible with the one of Table IV obtained using *Bochvar*'s internal based chain rule is not compatible with the distribution of Table II obtained using Kleene's based chain rule. □

A	B	$\pi^C_{B_I}(AB)$
a_1	b_1	0
a_2	b_1	1
a_3	b_1	1
a_1	b_2	1
a_2	b_2	0
a_3	b_2	1

Table V
COUNTER-EXAMPLE SHOWING THAT $\mathbb{F}_{\pi^C_{B_I}} \not\subseteq \mathbb{F}_{\pi^C_K}$.

In Bochvar's three-valued logic, the third value encodes ignorance and the internal conjunction operator is designed to capture the fact that *ignorance is contagious*. As a consequence, the obtained C-joint distribution contains more imprecision/conflict, hence $\mathbb{F}_{\pi^C_{B_I}}$ is a superset of $\mathbb{F}_{\pi^C_K}$.

Let us now move to Bochvar's external based chain rule. We define now the $\wedge^{B_E}$-based chain rule as follow: Let ω=$a_1a_2..a_n$, then

$$\pi_{B_E}(a_1, a_2, .., a_n) = \overset{B_E}{\min_{i=1..n}} (\theta^C_i(a_i|u_i)). \tag{8}$$

Example 3 (continued)

Let us now compute the joint distribution encoded by the network of Figure 2 using the min-based chain rule of Equation 8.

A	B	$\pi^C_{B_E}(AB)$
a_1	b_1	0
a_2	b_1	0
a_3	b_1	0
a_1	b_2	1
a_2	b_2	0
a_3	b_2	0

Table VI
JOINT DISTRIBUTION ENCODED BY THE NETWORK OF FIGURE 2 USING EQUATION 8.

The joint distribution obtained using the $\wedge^{B_E}$-based chain rule is a unique and boolean distribution and it is compatible with the one obtained using Equation 6. Hence, we have the following Proposition:

Proposition 5: Let $\mathbb{F}_{\pi^C_K}$ denote the set of boolean joint distributions π that are compatible with the C joint distribution π^C_K obtained using the $3V$-based chain rule of Equation 6. Let also π_{B_E} denote the boolean joint distribution obtained using the $\wedge^{B_E}$-based chain rule of Equation 8. Then,

$$\pi_{B_E} \in \mathbb{F}_{\pi^C_K}.$$

Proof. Let ω=$a_1a_2..a_n$ be an interpretation. In order to show that $\pi_{B_E} \in \mathbb{F}_{\pi^C_K}$, two cases are to be considered:

1) If $\pi_{B_E}(a_1a_2..a_n)$=0 then $\exists \theta^C_i \in \Theta^C$ such that $\theta^C_i(a_i|u_i)$=0 or $\theta^C_i(a_i|u_i)$=C. Hence, $\pi_K(a_1a_2..a_n)$=0 or $\pi_K(a_1a_2..a_n)$=C.
2) Otherwise $\pi_{B_E}(a_1a_2..a_n)$=1 requiring that $\forall \theta^C_i \in \Theta^C$, $\theta^C_i(a_i|u_i)$=1 ensuring that $\pi_K(a_1a_2..a_n)$=1. □

One can easily check that the distribution π_{B_E} is the least specific one that is compatible with π^C_K and it is obtained by replacing all the C values in the C-PN by 0.

In the following section, we address the question whether the well-known junction tree algorithm can be directly extended to the three-valued setting.

V. JUNCTION TREE ALGORITHM IN THE THREE-VALUED POSSIBILISTIC SETTING

The junction tree algorithm is a well-known and widely used inference algorithm in Bayesian networks [13]. The main idea of the junction tree algorithm is to decompose the joint belief distribution into a combination of local potentials (local joint distributions). In [6] an extension of the well-known junction tree algorithm to the three-valued possibilistic setting was proposed. In this section, we study whether this extension suits all the three-valued operators dealt with in the previous section. Namely, the question is to study whether the extended junction tree can factorize and recover a three-valued possibility distribution obtained using three-valued chain rules. Let us first recall the basic steps of this extension.

The graphical transformations (moralization and triangulation) are exactly the same as in the probabilistic version of the junction tree algorithm. Namely,

1) Moralization: In this step, a graphical transformation is performed on the initial directed DAG where the parents of each node are linked (married). After this step, the direction of the arcs is removed and the obtained graph is called the moralized graph.

2) Triangulation: In the moral graph, there may exist cycles having a length (number of edges) greater than three. The triangulation consists in adding edges to such cycles until every cycle has exactly three edges.

3) Initialization: In this step, the triangulated graph is compiled into a new data structure composed of clusters of nodes and separators. This structure is a new undirected graph where each node denotes a cluster of variables and separators denote the set of variables in common between two adjacent clusters. With each cluster or separator is associated a potential representing a kind of belief distribution regarding the variables involved in that cluster or separator. Initializing the clusters and separators is done as follows:

Let JT denote the junction graph obtained from the initial C-PN. In the following, we will use $\min^{3V}$ (resp. $\max^{3V}$) to denote a three valued min-based (resp. max-based) operator among $\min^{K}$ (resp. $\max^{K}$), $\min^{B_I}$ (resp. $\max^{B_I}$) and $\min^{B_E}$ (resp. $\max^{B_E}$).

- For each cluster $C_i{\in}JT$, initialize its C-based potential $\theta^C_{C_i}$ to 1 (namely, $\forall c_i{\in}D_{C_i}$, $\theta^C_{C_i}(c_i){\leftarrow}1$).
- For each separator $S_j{\in}JT$, initialize its C-based potential $\theta^C_{S_j}$ to 1 (namely, $\forall s_j{\in}D_{S_j}$, $\theta^C_{S_j}(s_j){\leftarrow}1$).
- For each variable $A_i{\in}V$, integrate its local C-based distribution $\theta^C_{A_k|U_k}$ into the cluster C_i (or the separator) containing A_k and its parents U_k. Namely,

$$\forall c_i{\in}D_{C_i},\ \theta^C_{C_i}(c_i){\leftarrow}\min^{3V}(\theta^C_{C_i}(c_i),\ \theta^C_{a_k|u_k}).$$

4) Stabilization: In order to guarantee that the marginal distribution relative to a given variable appearing in two adjacent clusters are the same, a stabilization operation consisting in propagating marginals is performed. Namely, the stabilization operation regarding two clusters C_i and C_j sharing the separator S_{ij} performs through two steps:

a) Collect evidence (separator update) : In this operation, each separator S_{ij} collects marginals from the clusters C_i and C_j sharing S_{ij}. This operation is done as follows:

$$\theta^C_{S_{ij}}(s_{ij}){\leftarrow}\min^{3V}(\theta^C_{C_i}(c_i/s_{ij}),\ \theta^C_{C_j}(c_j/s_{ij})),$$

where $\theta^C_{C_i}(c_i/s_{ij})$ (resp. $\theta^C_{C_j}(c_j/s_{ij})$ denotes the possibility degree of c_i (resp.c_j), a configuration of the variables involved in the cluster C_i (resp. C_j) without s_{ij}, a configuration of the separator S_{ij}. Note that the marginals are computed using the three-valued $\max^{3V}$ operator.

b) Distribute evidence (cluster update): Once the evidence is collected by a separator S_{ij}, it is distributed to the involved clusters as follows:

$$\theta^C_{C_i}(c_i){\leftarrow}\min^{3V}(\theta^C_{C_i}(c_i),\ \theta^C_{S_{ij}}(s_{ij})),$$
$$\theta^C_{C_j}(c_j){\leftarrow}\min^{3V}(\theta^C_{C_j}(c_j),\ \theta^C_{S_{ij}}(s_{ij})).$$

A. Junction tree algorithm based on Kleene's operators

It is shown in [6] that a C-based joint distribution obtained using the Kleenes' based chain rule of Equation 6 can be factorized using the $\min^K$ and $\max^K$ operators in the junction tree algorithm. Hence, we have the following proposition:

Proposition 6: Let C-PN be a three-valued based possibilistic network and JT_C=$<N$, $\Theta^C>$ be the junction tree obtained from the network C-PN where N denotes the set of clusters and separators and Θ^C=$\{\theta^C_1, .., \theta^C_m\}$ denotes the local C-based joint distributions associated with the clusters and separators. Then, for every variables' configuration $a_1, .., a_n$,

$$\pi^C(a_1, .., a_n){=}\min^K_{i=1..n}(\theta^C(a_i|u_i)){=}\min^K_{N_i\in N}(\theta^C_{N_i}(n_i)),$$

where n_i denotes the configuration of variables A_i involved in the node N_i (a node in a junction tree can be either a cluster or a separator). Proposition 6 states that the joint C-based distribution computed using the C-based chain rule of Equation 6 is equivalent to the one computed using the C-based junction tree. The proof of this proposition is based on the fact that the transformations of the initialization and stabilization steps don't change the C-based joint distribution. However, as it will be shown in the following, a C-based joint distribution obtained using the three-valued chain rule based on Bochvar's internal operators is not guaranteed to be directly factorized using the extended Junction tree algorithm.

B. Junction tree algorithm based on Bochvar's operators

Regarding Bochvar's operators, it is obvious that for the external operators, the extension of the junction tree algorithm correctly factorizes and recovers the C-based joint distribution obtained using Equation 8 as stated in the following proposition.

Proposition 7: Let C-PN be a three-valued based possibilistic network and JT_C=$<N$, $\Theta^C>$ be the junction tree obtained from the network C-PN where N denotes the set of clusters and separators and Θ^C=$\{\theta^C_1, .., \theta^C_m\}$ denotes the local C-based joint distributions associated with the clusters and separators. Then, for every variables' configuration $a_1, .., a_n$,

$$\min^{B_E}_{i=1..n}(\theta^C(a_i|u_i)){=}\min^{B_E}_{N_i\in N}(\theta^C_{N_i}(n_i)),$$

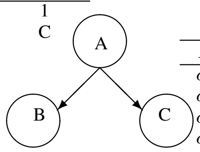
The result of Proposition 7 is straightforward since the obtained distribution using Bochvar's external operator is a boolean possibility distribution. Now, using the internal operators, it is not guaranteed that the factorized C-based distribution is equivalent to the one obtained using Equation 7 as it is shown in the following counter-example.

Example 4

Let us illustrate this on the network of Figure 4.

A	$\theta^C(A)$
a_1	1
a_2	C

B	A	$\theta^C(B\|A)$
b_1	a_1	1
b_2	a_1	C
b_1	a_2	0
b_2	a_2	1



B	A	$\theta^C(C\|A)$
c_1	a_1	1
c_2	a_1	0
c_1	a_2	0
c_2	a_2	1

Figure 4. Example of a C-PN over three binary variables A, B and C.

The corresponding junction tree graph after the initialization step is given in Figure 5.

B	A	$\theta(AB)$
b_1	a_1	1
b_2	a_1	C
b_1	a_2	0
b_2	a_2	1

A	$\theta(A)$
a_1	1
a_2	C

C	A	$\theta(AC)$
c_1	a_1	1
c_2	a_1	0
c_1	a_2	0
c_2	a_2	1

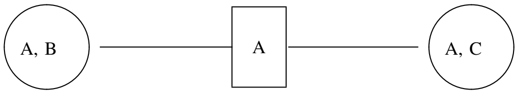


Figure 5. Junction tree obtained after the initialization step from the network of Figure 4.

After the stabilization step, we obtain the junction tree of Figure 6.

B	A	$\theta(AB)$
b_1	a_1	C
b_2	a_1	C
b_1	a_2	C
b_2	a_2	C

A	$\theta(A)$
a_1	C
a_2	C

C	A	$\theta(AC)$
c_1	a_1	C
c_2	a_1	C
c_1	a_2	C
c_2	a_2	C

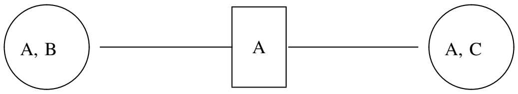


Figure 6. Junction tree obtained after the stabilization step from the network of Figure 4.

It is easy to see that using the junction of Figure 6, for every interpretation ω=$a_ib_jc_k$, *we have* $\pi^{B_I}_{JT}(\omega)$=*C. However, using the Bochvar's internal chain rule of Equarion 7,* $\pi_{B_I}(a_1b_1c_1)$=*1 and* $\pi_{B_I}(a_1b_1c_2)$=*0.*

This result is not surprising since Bochvar's internal connectors are designed to make contagious ignorance (in our case, imprecision/conflict). This is due to the fact that during the stabilization step, collecting and distributing evidence contaminates the values 0 and 1 by the C values. It is worth pointing out that for the other connectors, the C-based joint distribution can be factorized with the extended junction tree algorithm without any loss.

VI. CONCLUSION

This paper dealt with special kinds of possibilistic networks suitable for encoding both uncertainty and imprecise/conflicting beliefs. More precisely, the paper proposed and analyzed several semantics for three-valued possibilistic networks and provided precise relations relating the different semantics. We proposed two categories of semantics where one consists in viewing a possibilistic network as a family of compatible networks while the second is based on deriving a three-valued joint distribution by extending the chain rule to the three-valued setting. We showed that none of the studied three-valued conjunction operators recovers the semantics of C-PNs viewed as families of compatible networks. We addressed inference in three-valued possibilistic networks and showed that the direct extension of the junction tree algorithm to the three-valued setting guarantees a correct factorization of a C-joint distribution when using Kleene and Bochvar's external connectors but the situation is different when using Bochvar's internal connectors. Moreover, this extension does not induce any extra computational complexity in comparison to the standard possibilistic setting.

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