Université Paris Descartes, BME-Paris Master - UE Math II - 2016-2017
Math test

Exercise 1 We consider the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$f(x_1, x_2) = 2x_1^4 e^{x_2} - x_2^2 + x_2.$$

1. Show that f has only one critical point and compute it.
2. Write the hessian of f as a function of $x = (x_1, x_2)$ and then at the critical point. Does f have a global minimum over $\mathbb{R}^2$?
3. Let g be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(t) = f(t, 1/2)$. Show that g has a unique global minimum over $\mathbb{R}$. Deduce the type of the critical point of f : local minimum, local maximum, or saddle point.

Exercise 2

1. Recall the definition of a convex function $u : \mathbb{R}^n \to \mathbb{R}$.
2. Using the definition, show that if two functions $u, v : \mathbb{R}^n \to \mathbb{R}$ are convex, then the function w defined by $w(x) = \max(u(x), v(x))$ is also convex.

Exercise 3 We consider the following simple model for the World Wide Web : it is composed of N webpages, and each webpage contains links to some other webpages. A user starts by choosing one of the N webpages at random. Then at each step, he either : a) (with probability p) : clicks on one of the links at random on the webpage, or b) (with probability $1-p$) : moves to another webpage chosen at random among the $N-1$ others. As an example, we consider the following case with $N = 3$ and pages labelled $1, 2, 3$:

— page 1 contains a link to page 2 only,
— page 2 contains links to pages 1 and 3,
— page 3 contains a link to itself only.

We denote X_k the label of the webpage and $\pi(k)$ the vector of probabilities at step k :

$$\pi(k) = (P(X_k = 1), P(X_k = 2), P(X_k = 3)).$$

We first consider the case $p = 1$: the user always follows one of the links on the current webpage.

1. Write the transition probabilities $P(X_{k+1} = j | X_k = i)$ for all states i, j, the transition matrix T, and the diagram of the Markov chain.
2. What are the probabilities the user will be on page 1, on page 2, on page 3, after two steps (i.e. for the third webpage visited) ?
3. Give the classes of this Markov chain, and there types : transient or recurrent, period. What is the behaviour of the Markov chain when $k \to \infty$? Does $\pi(k)$ converge to some limit ?

We now consider the case $p < 1$.

4. Write the transition probabilities of the Markov chain as functions of p.
5. Now we fix $p = 2/3$. Write the transition matrix and the diagram of the Markov chain. Explain why this Markov chain has only one class and is aperiodic. Does $\pi(k)$ converge to some limit ?