Université Paris Descartes, BME-Paris Master - UE Math II - 2013-2014
Math test

Exercise 1 Let $p^1 = (p_1^1, p_2^1)$, $p^2 = (p_1^2, p_2^2)$, ..., $p^n = (p_1^n, p_2^n)$ be n points in $\mathbb{R}^2$. We want to find the point that minimize the sum of squared distances to all points p^i : hence, for any point $x = (x_1, x_2)$ we define the following functional :

$$J(x) = \sum_{i=1}^{n} \|x - p^i\|^2.$$

1. Calculate $\frac{\partial J}{\partial x_1}(x)$ and $\frac{\partial J}{\partial x_2}(x)$, and deduce that J has a unique critical point x^* in $\mathbb{R}^2$. How is called this point in geometrical terms ?
2. Show that x^* is a local minimum point of J.
3. Explain why x^* is also the unique global minimum point of J.

Exercise 2 We consider the following toy example of birth and death process. We place some bacteria in a growth medium. We assume that the amount of nutriments in the growth medium is given by a constant value F, and we suppose the following rules for the evolution of the population of bacteria between one step and the next :

- if the number of bacteria is lower or equal to F, **each** bacteria may divide with probability $1/2$, and no bacteria will die.
- if the number of bacteria is above F, no bacteria can divide, and there is probability $1/2$ that **one** bacteria will die. Only one bacteria can die at each step.

1. We suppose that $F = 2$ and that we place one bacteria at first in the medium : $X_1 = 1$, where X_k denotes the number of bacteria at step k. Express $P(X_{k+1} = 1), P(X_{k+1} = 2), \ldots$ in terms of $P(X_k = 1), P(X_k = 2), \ldots$ and explain why the evolution of the population is a Markov chain process. Write its diagram and its transition matrix. Using Matlab, find the limit probability state of the Markov chain. Calculate $E(X_k)$ for large k (average number of bacteria at steady-state)
2. We suppose now that $F = 3$. Write again the diagram, the transition matrix, and find the limit probability state.