PDE in image processing

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Outline

- *Motivations*
- *Linear Denoising*
- *Non linear PDEs*
- *Variational Methods*
- *Axiomatic Approach*
- *Morphological Operators*
- *Geometrical Schemes*

Motivations

An image is a function $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^d$.

Problem: Restoration and/or denoising of an image before «high-level », operations *e.g.* contour extraction , segmentation

Idea: The observed image u_0 depends on the “real” u:

$$u_0 = Au + n\,,$$

with A a linear (e.g. convolution) or non linear operator and n additive noise;

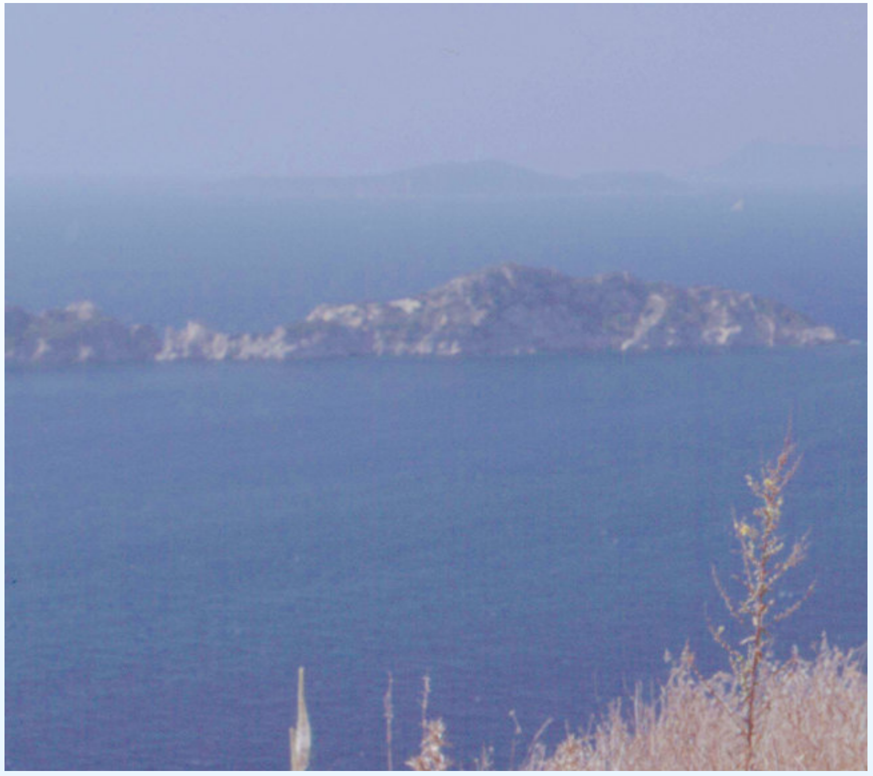
If the noise is purely additive ($A = Id$) one might smooth the image as proposed in classical signal processing by using low-pass filters.

Application

noisy image

denoised image

Application (cont.)



original image

denoised image

Linear denoising

Classical example: the Gaussian Filter,

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2}\, e^{-\dfrac{|x|^2}{2\sigma^2}}\,, \qquad \text{with } x = (x_1, x_2) \in \mathbb{R}^2\,,$$

then, by convolution with $u_0 = u + n$,

$$(G_\sigma \star u_0)(x) = \int_{\mathbb{R}^2} G_\sigma(x-y)u_0(y)\,dy\,.$$

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- The parameter $\sigma > 0$ determines the spatial size of the details which are “eliminated” by this filter: the bigger σ, the smoother the result, the lesser details are kept.
- Convolution is efficiently computed using the FFT.

Linear denoising (cont.)

Now convolution with the Gaussian amounts to solve the heat equation:

$$\begin{cases} u_t(x,t) & = & \Delta u(x,t) \\ u(x,0) & = & u_0(x)\,. \end{cases}$$

- Thus regularization through convolution is replaced by isotropic diffusion with $t = \sigma^2/2$.

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- Example of a linear scale space operator $T_t u_0 = u(.,t)$.

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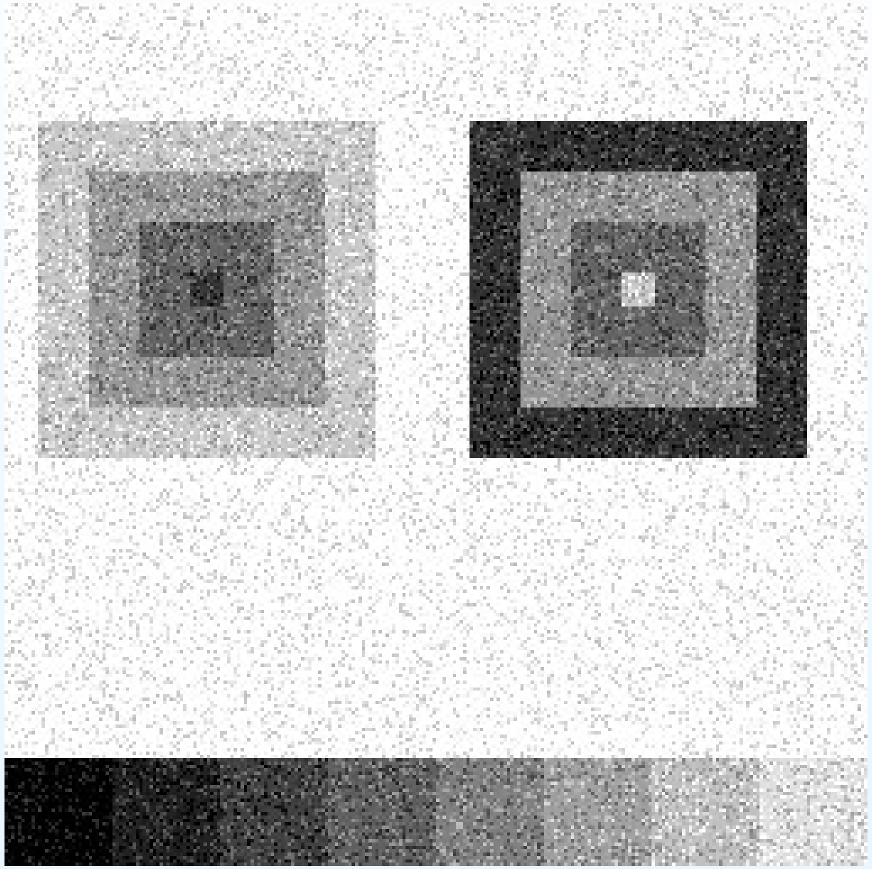
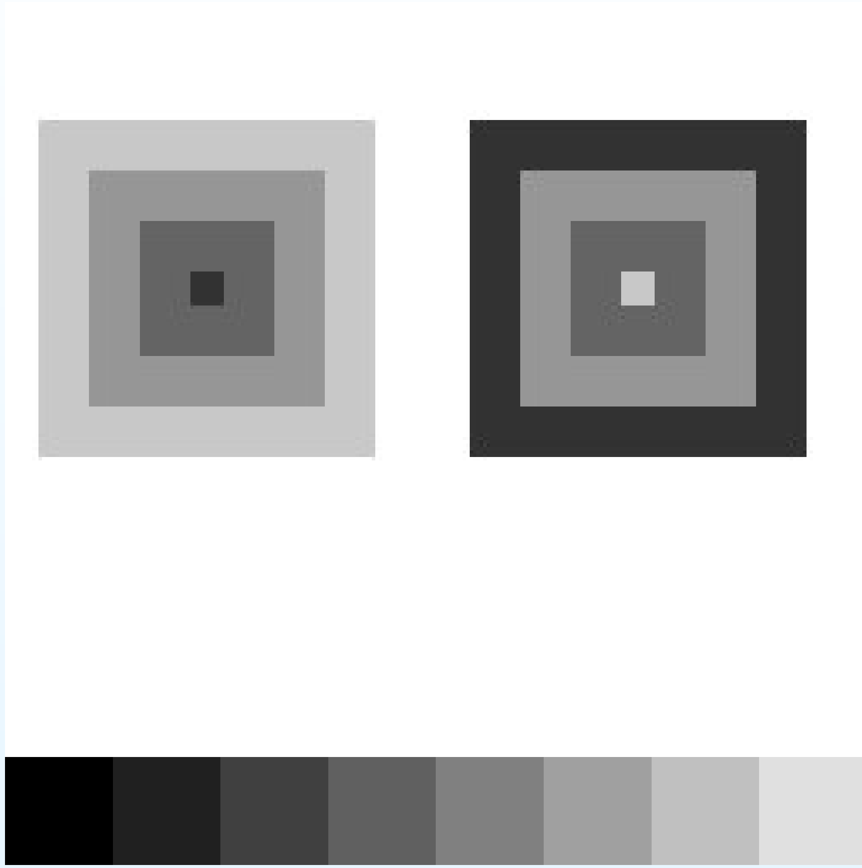
Conclusion: Isotropic diffusion operates like a weighted mean filter, eliminating noise but blurring contours.

Example

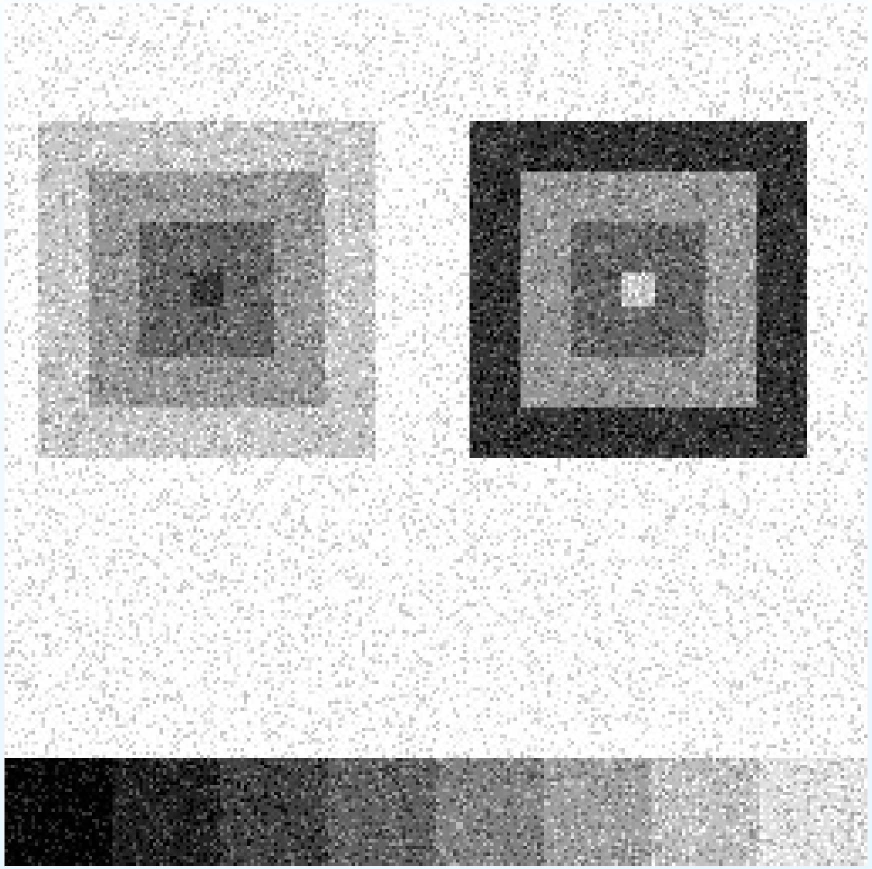


$$u_0 = u + n = u(.,0)$$



$$u$$

Example



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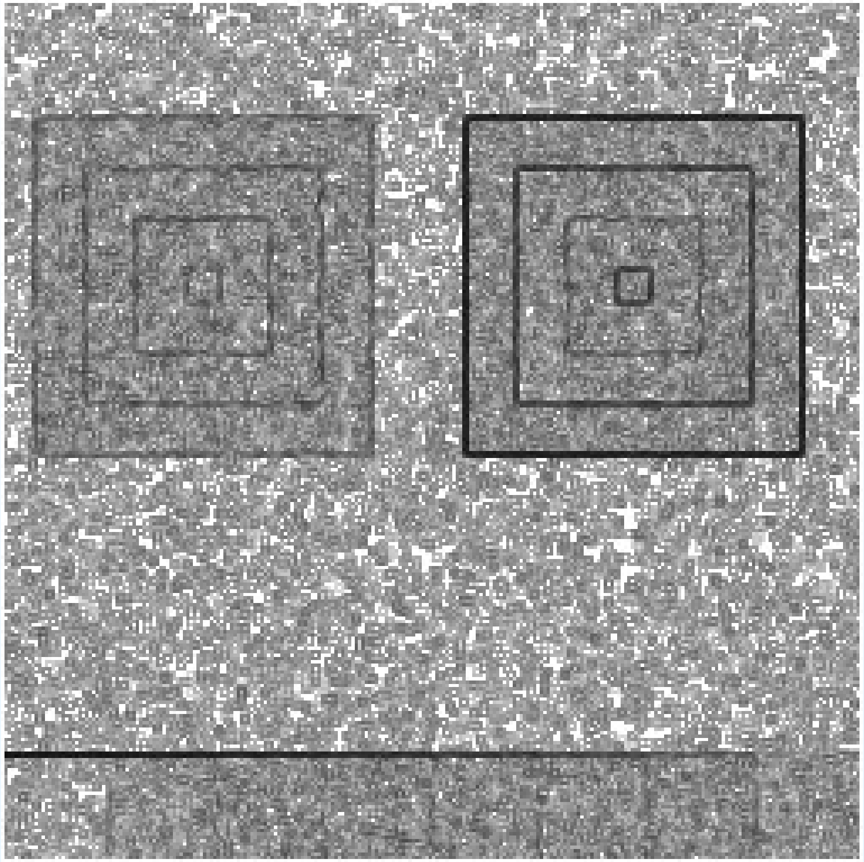
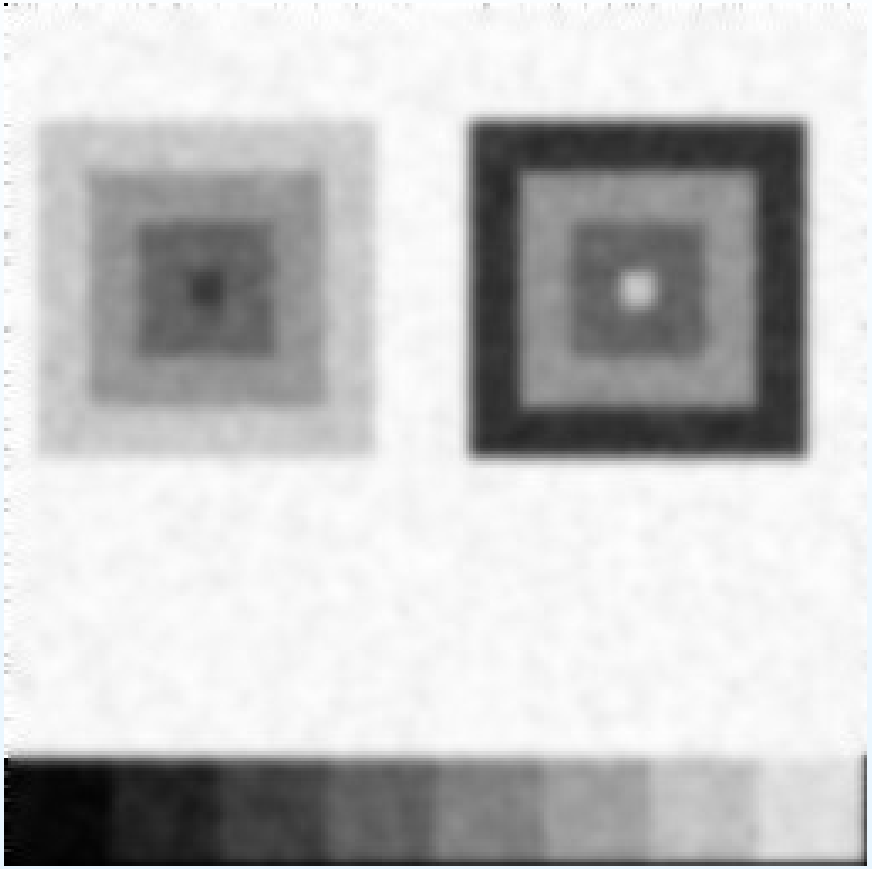


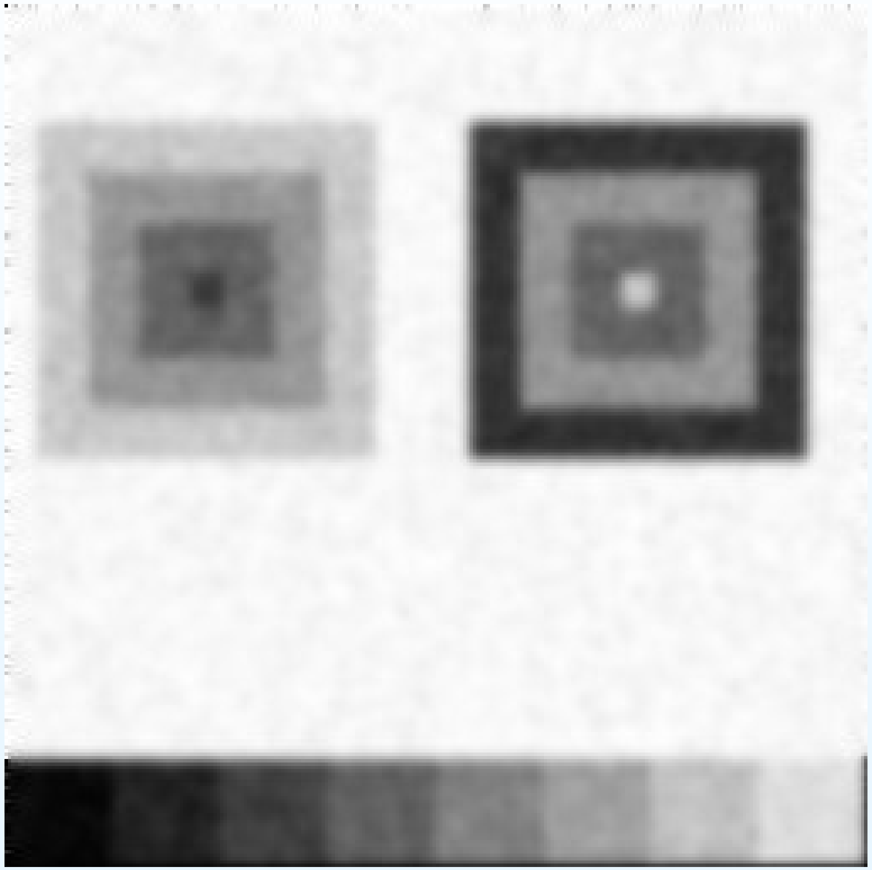
$$\|\nabla u_0\|$$

Example (cont.)



$$G_\sigma \star u_0 = u(., \sigma^2/2)$$

Example (cont.)



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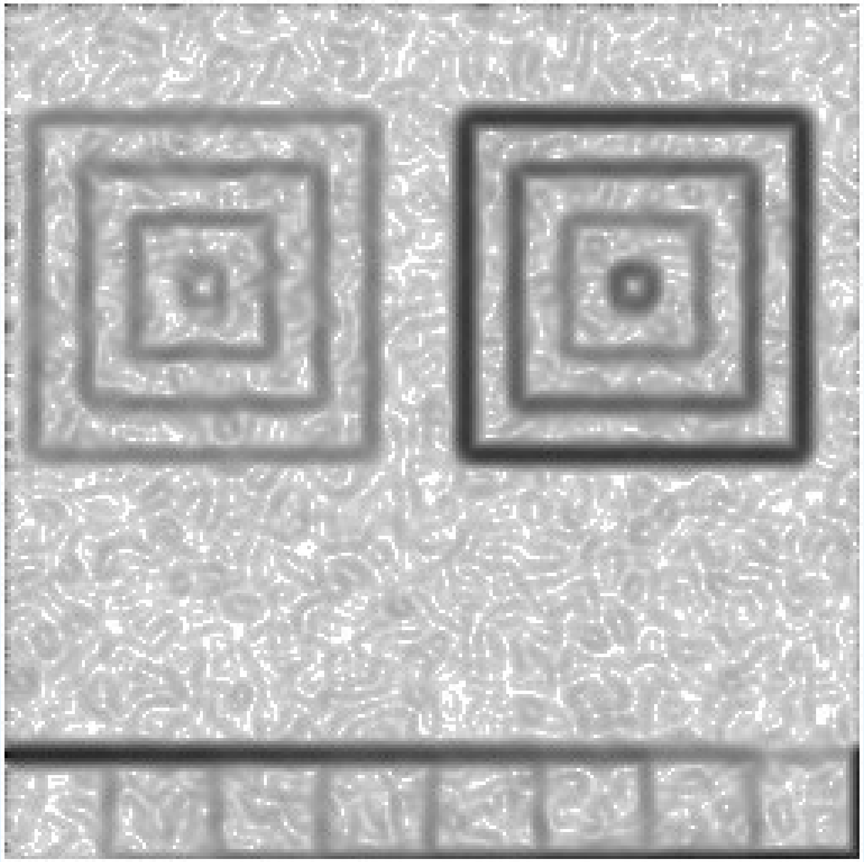
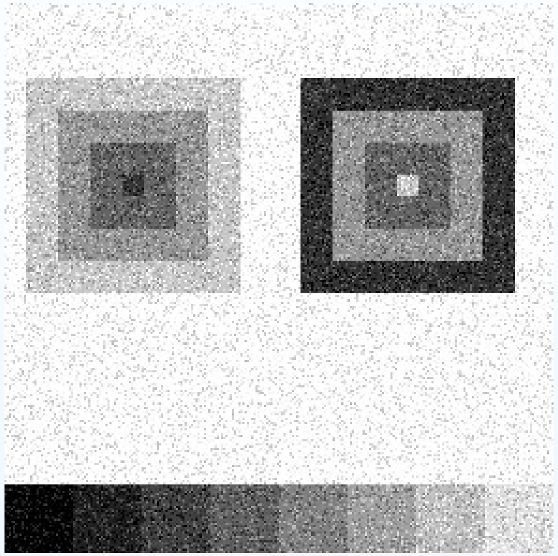
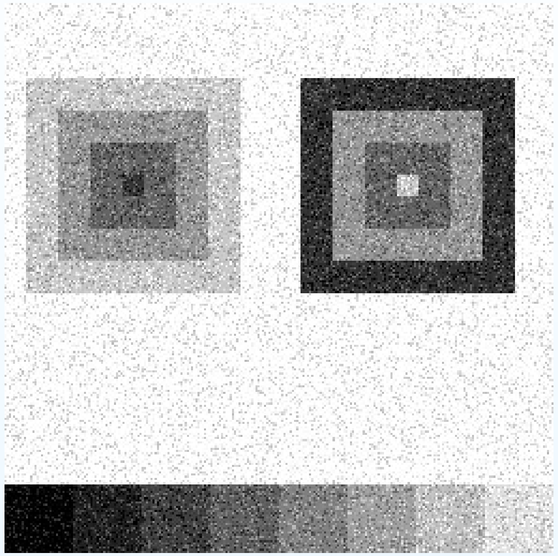


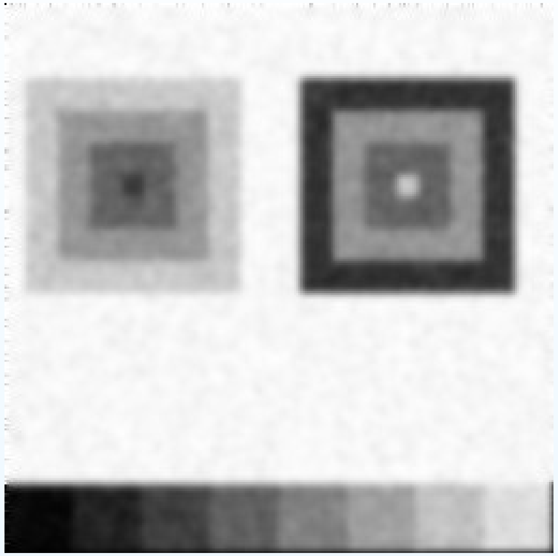
$$\|\nabla u(.,\sigma^2/2)\|$$

Linear scale space, for $t = 0$

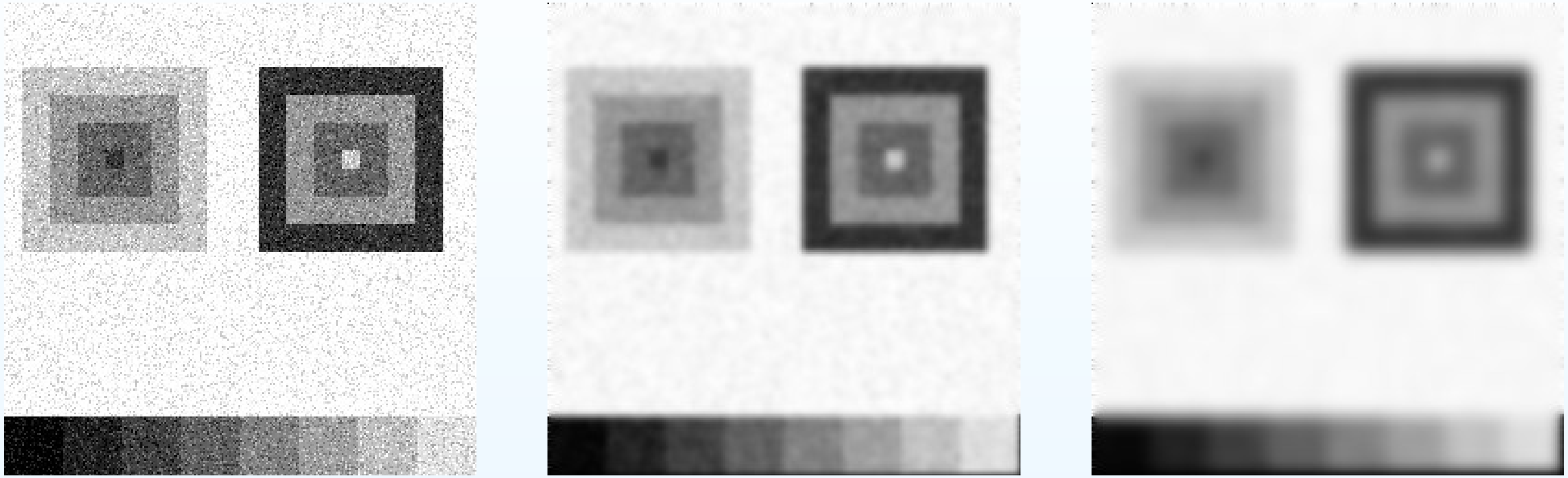


Linear scale space, for $t = 0$,20

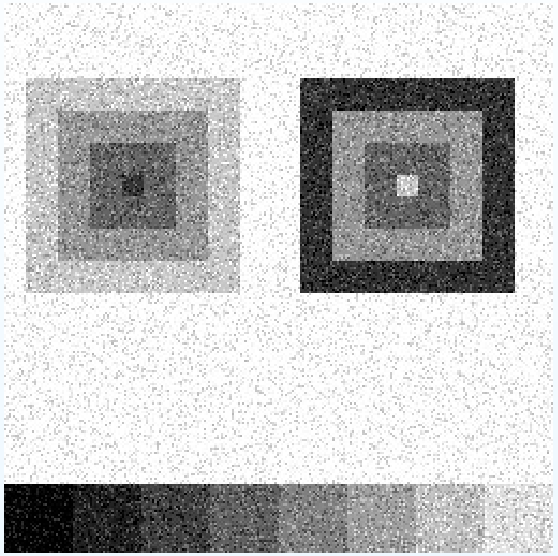
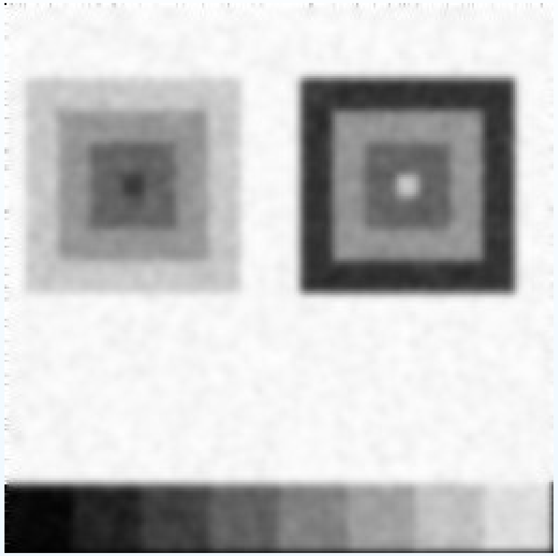
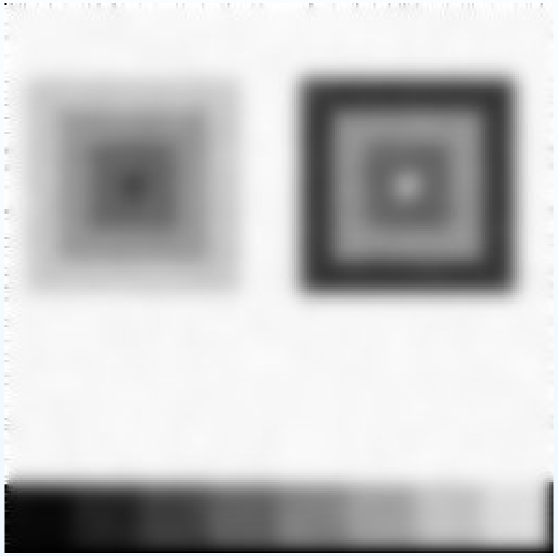




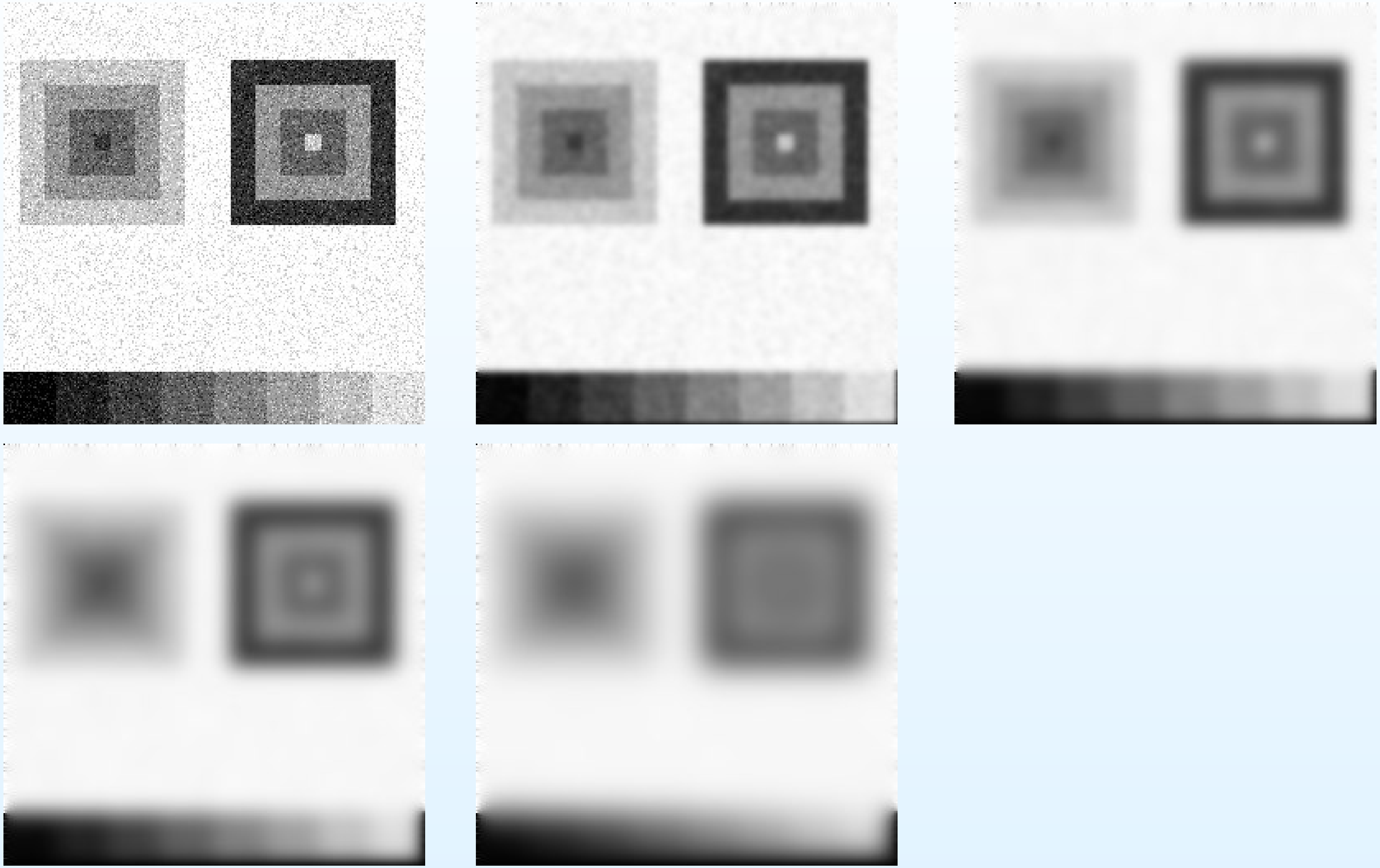
Linear scale space, for $t = 0$,20 ,100



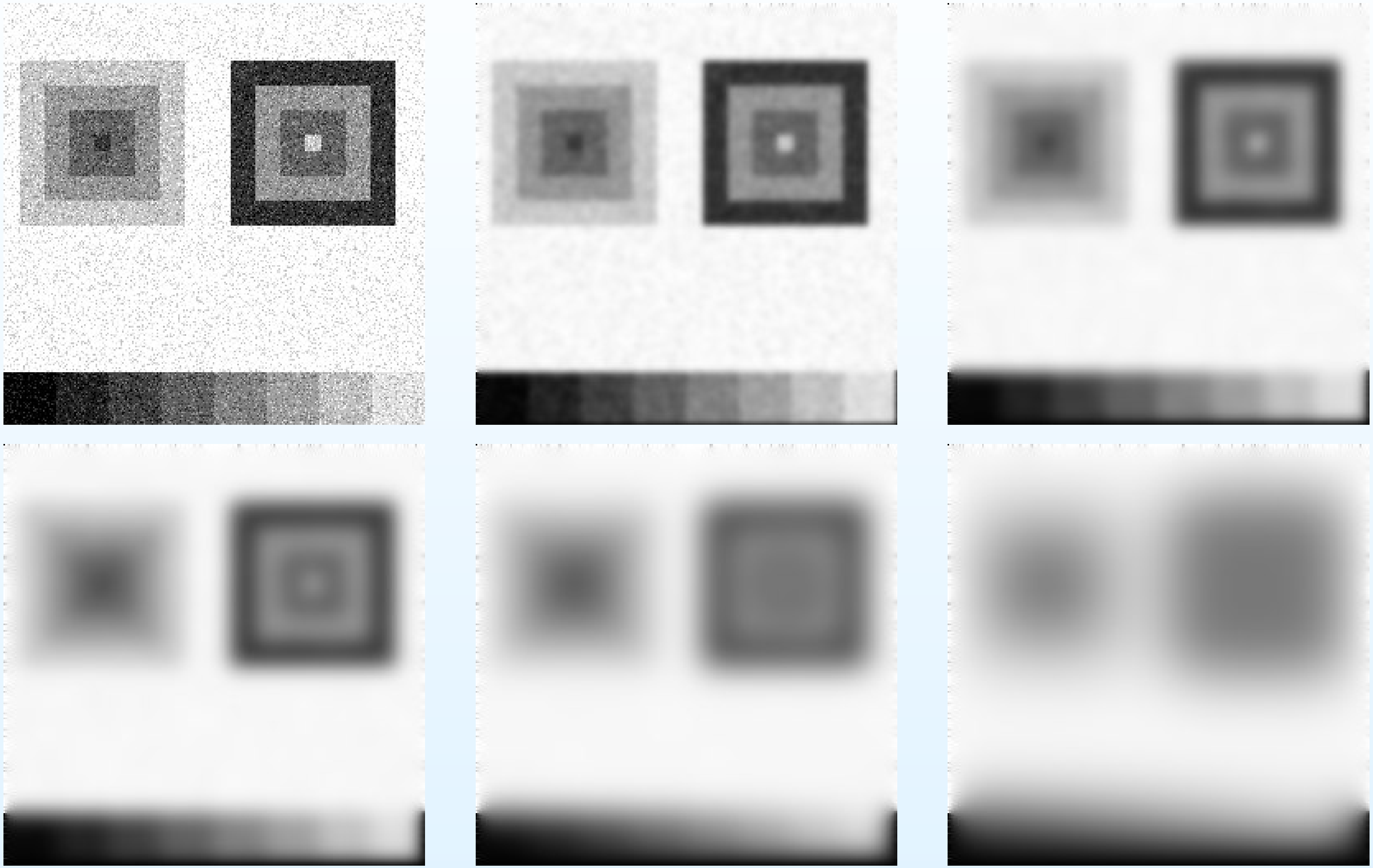
Linear scale space, for $t = 0$,20 ,100 ,200


Linear scale space, for $t = 0\ ,20\ ,100\ ,200\ ,500$



Linear scale space, for $t = 0\ ,20\ ,100\ ,200\ ,500\ ,2000$



Model of MALIK and PERONA (1987)

Idea: No diffusion across the boundaries, *i.e.* diffuse only in the direction of the gradient ∇u when $\|\nabla u\|$ is large.

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The proposed PDE writes:

$$\frac{\partial u}{\partial t} = \text{div}\Big(g(|\nabla u|)\,\nabla u\Big) \qquad \text{with } u(x,0) = u_0(x)\,,$$

with $g(r)$, $r \geq 0$, a non increasing function and $g(0) = 1$.
An often used diffusion coefficient is

$$g(r) = \frac{1}{1 + (r/\lambda)^2}\,, \quad \lambda \in \mathbb{R}_+^*\,.$$

Interpretation

Suppose u is a smooth function and let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$, then

$$H_u(x,y) = \begin{pmatrix} u_{xx}(x,y) & u_{xy}(x,y) \\ u_{xy}(x,y) & u_{yy}(x,y) \end{pmatrix}$$ is the *Hessian matrix*

and $$\Delta u(x,y) = \mathrm{Tr}(H_u(x,y))\,,\ \nabla u(x,y) = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix}.$$

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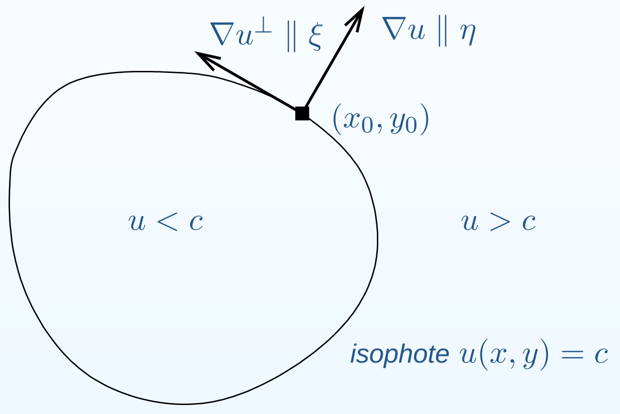
For η, ξ in $\mathbb{R}^2$ we write

$$u_\eta \stackrel{\Delta}{=} \frac{\mathrm{d}u}{\mathrm{d}\eta} = \nabla u^t\,\eta \quad \text{and} \quad u_{\eta\xi} \stackrel{\Delta}{=} d^2u(\eta,\xi) = \eta^t\,H_u\,\xi\ .$$

Interpretation (cont.)

The set $I_c = \{(x,y) \,/\, u(x,y) = c\}$ is an *isophote* :



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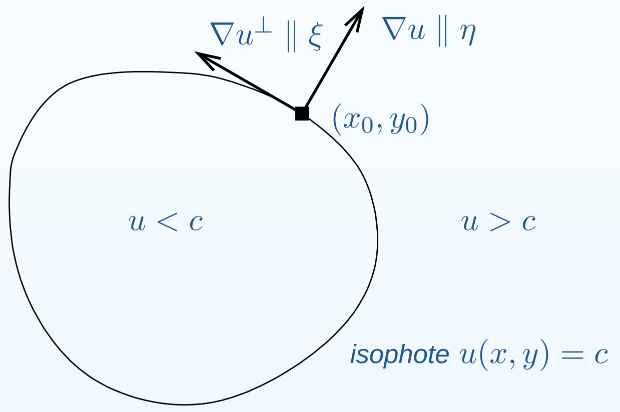

In $(x_0, y_0) \in I_c$ define the local frame

$$(\eta, \xi)_{|(x_0,y_0)} = \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u^\perp}{|\nabla u|} \right)_{|(x_0,y_0)}$$

Then $\quad u_\xi = \nabla u^t \, \xi = 0 \quad$ and $\quad u_\eta = \nabla u^t \, \eta = |\nabla u|$.

Interpretation (cont.)

In each $(x, y) \in I_c$:

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{1}{|\nabla u|}\left(\Delta u - d^2u\,(\eta,\eta)\right) = \frac{1}{|\nabla u|}\,d^2u\,(\xi,\xi)\ .$$

and $$\Delta u = d^2u(\eta,\eta) + d^2u(\xi,\xi) = u_{\eta\eta} + u_{\xi\xi}\,.$$

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The linear isotropic heat equation writes

$$u_t = u_{\eta\eta} + u_{\xi\xi}\ .$$

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The MALIK-PERONA model writes

$$u_t = (g(u_\eta) + u_\eta g'(u_\eta))u_{\eta\eta} + g(u_\eta)u_{\xi\xi}\ .$$

Interpretation of the MALIK-PERONA model

$$u_t = G'(u_\eta)u_{\eta\eta} + g(u_\eta)u_{\xi\xi} \ ,$$

where $G(r) = rg(r)$ and $G'(r) = g(r) + rg'(r)$.

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A limiting case is the total variation minimization pde :

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\ ,$$

here $g(r) = 1/r$, we recall that for u regular $TV(u) = \int_\Omega \|\nabla u\|$.

Variational approach

Idea: For $u_0 = Au + n$ characterize u to be the minimum of

$$E(u) = \frac{1}{2}\|u_0 - Au\|^2 + \lambda \int_\Omega \Phi(|\nabla u|)dx \ .$$

i.e. data fidelity term $+$ regularization

Here λ is a “scale” parameter.

Most common examples:

- $\Phi(|\nabla u|) = |\nabla u|^2/2\,;$
- $\Phi(|\nabla u|) = |\nabla u|\,.$

Variational approach (cont.)

Compute the first variation δE of E and write a *gradient descent*:

$$u_t = A^*(u_0 - Au) + \lambda \mathrm{div}\left(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)$$

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If $\Phi(|\nabla u|) = |\nabla u|^2/2$, L^2 regularization

$$u_t = A^*(u_0 - Au) + \Delta u\,.$$

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If $\Phi(|\nabla u|) = |\nabla u|$, Total Variation regularization

$$u_t = A^*(u_0 - Au) + u_{\xi\xi} \, .$$

diffusion perpendicular to the gradient

Axiomatic approach

Idea: Formalize the properties of multiscale filters T_t.

- **[Causality]** $T_0 = Id$ and $\forall s < t, \exists\, T_{s,t} :\ T_t = T_{s,t} \circ T_s\,;$
  $\Longrightarrow$ no creation of information

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- **[Local Maximum]** If $u(x) > v(x)$ in $B(x_0, r) \setminus \{x_0\}$, then, for all t and for h small enough: $(T_{t,t+h}u)(x_0) \geq (T_{t,t+h}v)(x_0)$;
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- **[Regularity]** For $Q(x) = \frac{1}{2}(x - x_0)^t A(x - x_0) + p^t(x - x_0) + c$, with $A^t = A$, $p \in \mathbb{R}^2$ and $c \in \mathbb{R}$ then there exists F:
  $$\frac{\partial(T_{t,t+h}Q)}{\partial h}(x_0) = F(A, p, c, x_0, t),$$
  with F continuous everywhere, except $p = 0$.
  $\Longrightarrow$ locally second order characteristics rule over T_t

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We call these the **[Natural]** properties of a filter.

Axiomatic approach (cont.)

Other desirable invariance properties of the filters T_t:

- **[Translation Invariance]** For $(\tau_\alpha u)(x) = u(x + \alpha)$:
  $T_t(\tau_\alpha u) = \tau_\alpha(T_t u)$;

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- **[Affine Invariance]** For A an invertible affine transform, with $(Au)(x) = u(Ax)$:

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  where is λ increasing in t (zooms influence the filter scale).

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  where is λ increasing in t (zooms influence the filter scale).
- **[Contrast invariance]** For H continuous, nondecreasing:
  $$T_t(H(u)) = H(T_t(u))\,;$$

Axiomatic approach: results

Definition: A family of operators $(T_t)_t)$ is called morphological if **[Translation Invariance]** and **[Contrast invariance]** hold.

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All the **[Natural]**, **[Isotropic Invariant]** and **morphological** filters verify:

$$u_t = |\nabla u| G\,(curv(u), t)$$

with G a continuous, decreasing function w.r. to

$$\begin{aligned} curv(u) &= \mathrm{div}(\nabla u / |\nabla u|) \\ &= \text{curvature of the isophote } I_c\,. \end{aligned}$$

Axiomatic approach: results

Definition: A family of operators $(T_t)_t)$ is called morphological if **[Translation Invariance]** and **[Contrast invariance]** hold.

All the **[Natural]**, **[Isotropic Invariant]** and **morphological** filters verify:

$$u_t = |\nabla u| G\left(curv(u), t\right)$$

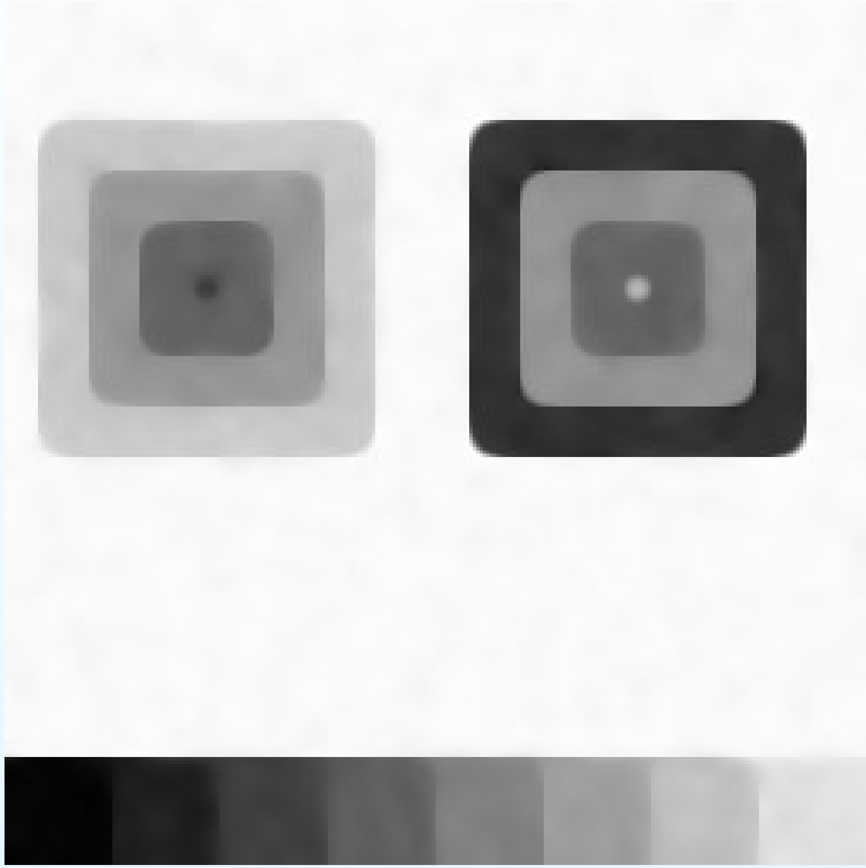
with G a continuous, decreasing function w.r. to

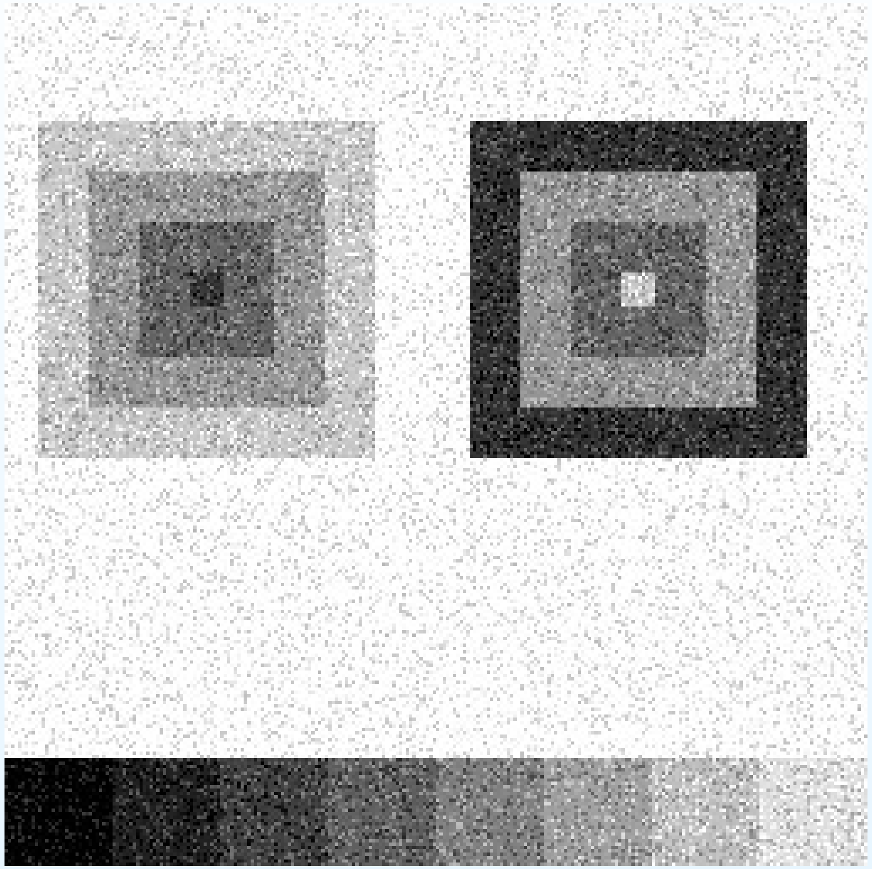
$$\begin{aligned} curv(u) &= \operatorname{div}(\nabla u / |\nabla u|) \\ &= \text{curvature of the isophote } I_c\,. \end{aligned}$$

If $G(curv(u), t) = curv(u)$ we obtain the mean curvature motion :

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = u_{\xi\xi}$$

Example of mean curvature motion

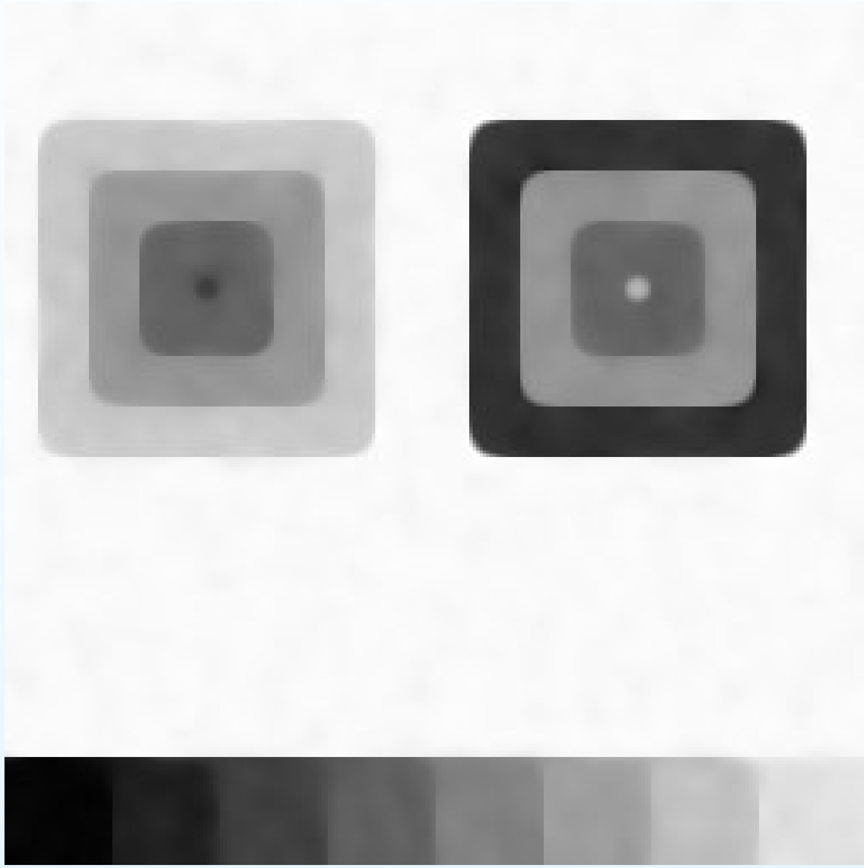


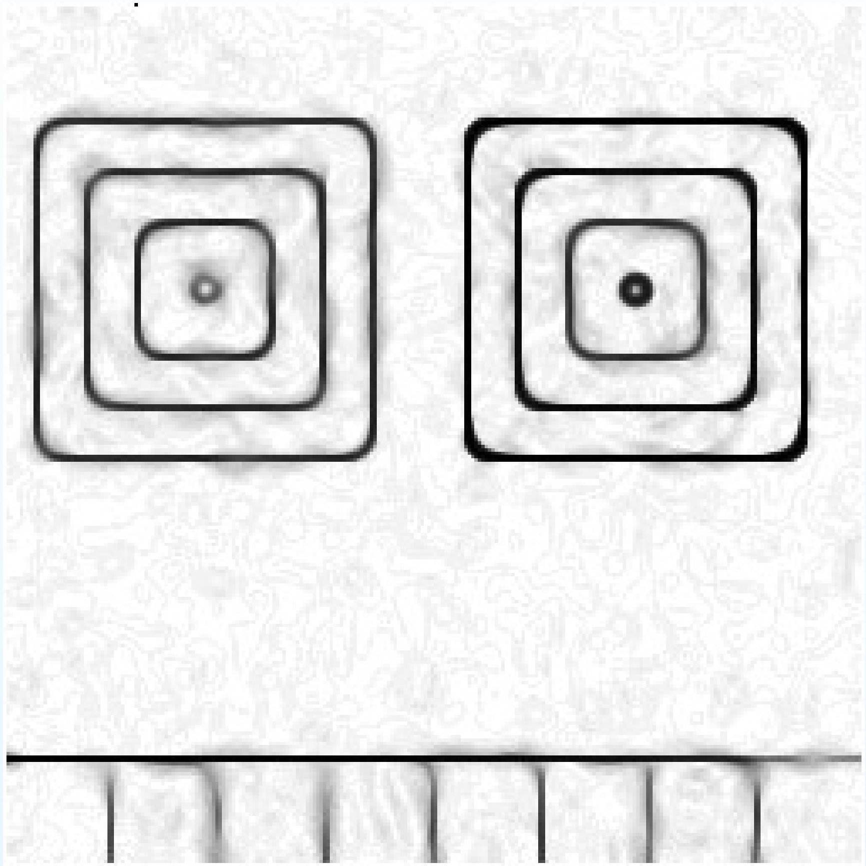


$$\frac{\partial u}{\partial t} = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = u_{\xi\xi}$$

original

Example of mean curvature motion





$$\frac{\partial u}{\partial t} = |\nabla u| \,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = u_{\xi\xi}$$

$$\|\nabla u\|$$

Axiomatic approach: results (cont.)

If we replace **[Isotropic Invariance]** by **[Affine Invariance]**
a unique pde is obtained

$$u_t = |\nabla u|\, curv(u)^{1/3} = (u_y^2 u_{xx} + u_x^2 u_{yy} - 2u_x u_y u_{xy})^{1/3} \, .$$

Thus there exists a unique affine morphological scale space.

Axiomatic approach: results (cont.)

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- In the preceding u is a weak solution in the sense of *viscosity solutions*.
- It is difficult/impossible to obtain numerical schemes verifying all the desired properties, *e.g.*
  - rotation invariance on a rectangular grid;
  - contrast invariance with finite difference schemes;

Morphological operators

Idea: Mathematical morphology shows that monotone increasing and translation invariant operators acting on continuous functions can be written:

$$(T_t u)(x) = \inf_{B \in \mathcal{B}_t} \sup_{y \in x+B} u(y) \qquad (1)$$

or

$$(T_t u)(x) = \sup_{B \in \mathcal{B}_t} \inf_{y \in x+B} u(y) \qquad (2)$$

with $\mathcal{B}_t$, the sets of structuring elements, a family of convex subsets of $\mathbb{R}^2$.

For $\mathcal{B}_t = B(O,t)$, (1) defines a dilation and (2) an erosion.

Morphological operators (cont.)

Theorem: Consider the morphological operator C_t defined by

$$(C_t u)(x) = \frac{1}{2}\left(\inf_{\theta \in [0,\pi[} \sup_{x+S_{(\theta,2t)}} u(y) + \sup_{\theta \in [0,\pi[} \inf_{x+S_{(\theta,2t)}} u(y) \right)$$

with $S_{(\theta,2t)}$ a segment of center 0, length $4\sqrt{t}$ and direction $\theta \in [0,\pi[$. Then

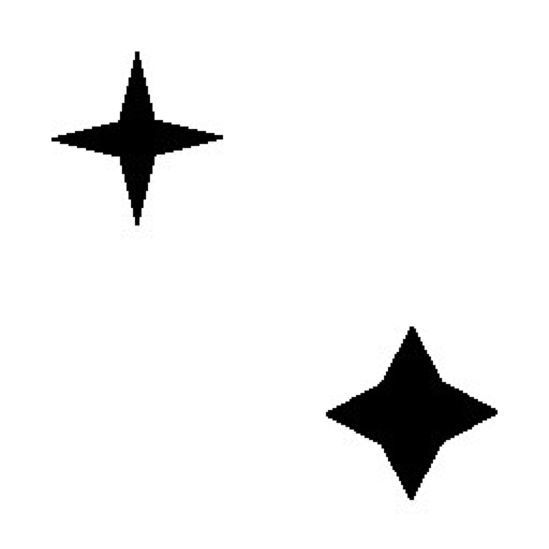
$$\lim_{m\to\infty} C^m_{\frac{t}{m}} u = u$$

uniformly in t, to the viscosity solution of the *mean curvature motion*

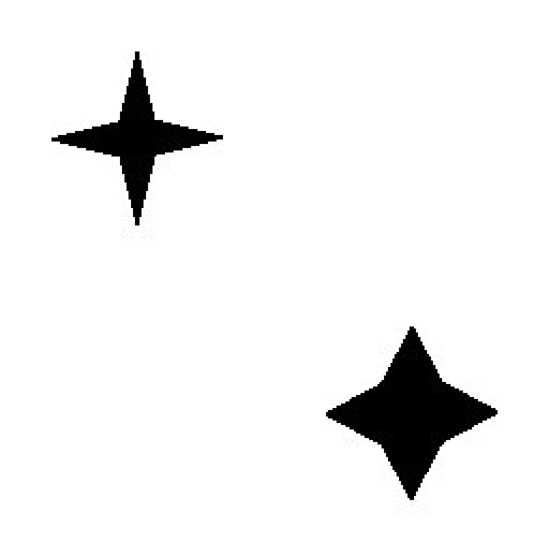
$$u_t = |\nabla u| \, curv(u) \ .$$

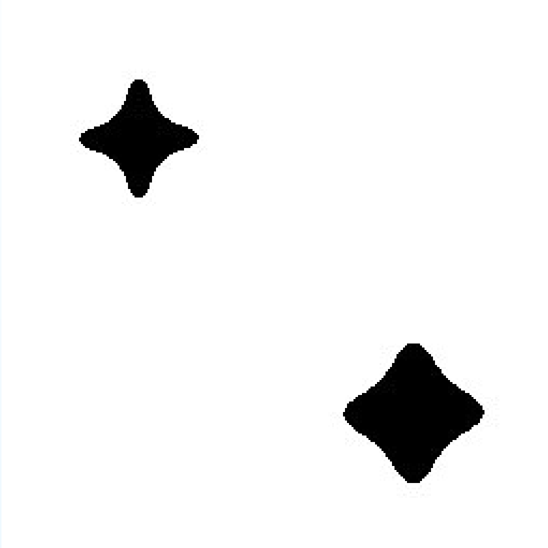
Thus by iterating morphological operators we obtain $u(.,t)$.

MCM scale space, for 0 iterations

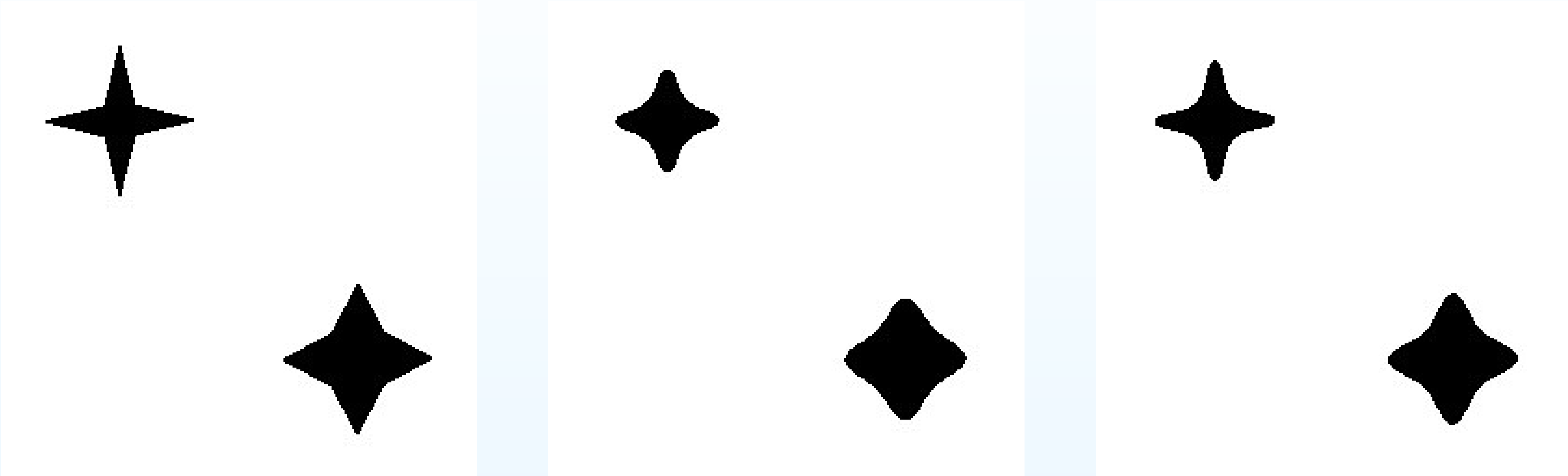


MCM scale space, for 0 , 20 iterations

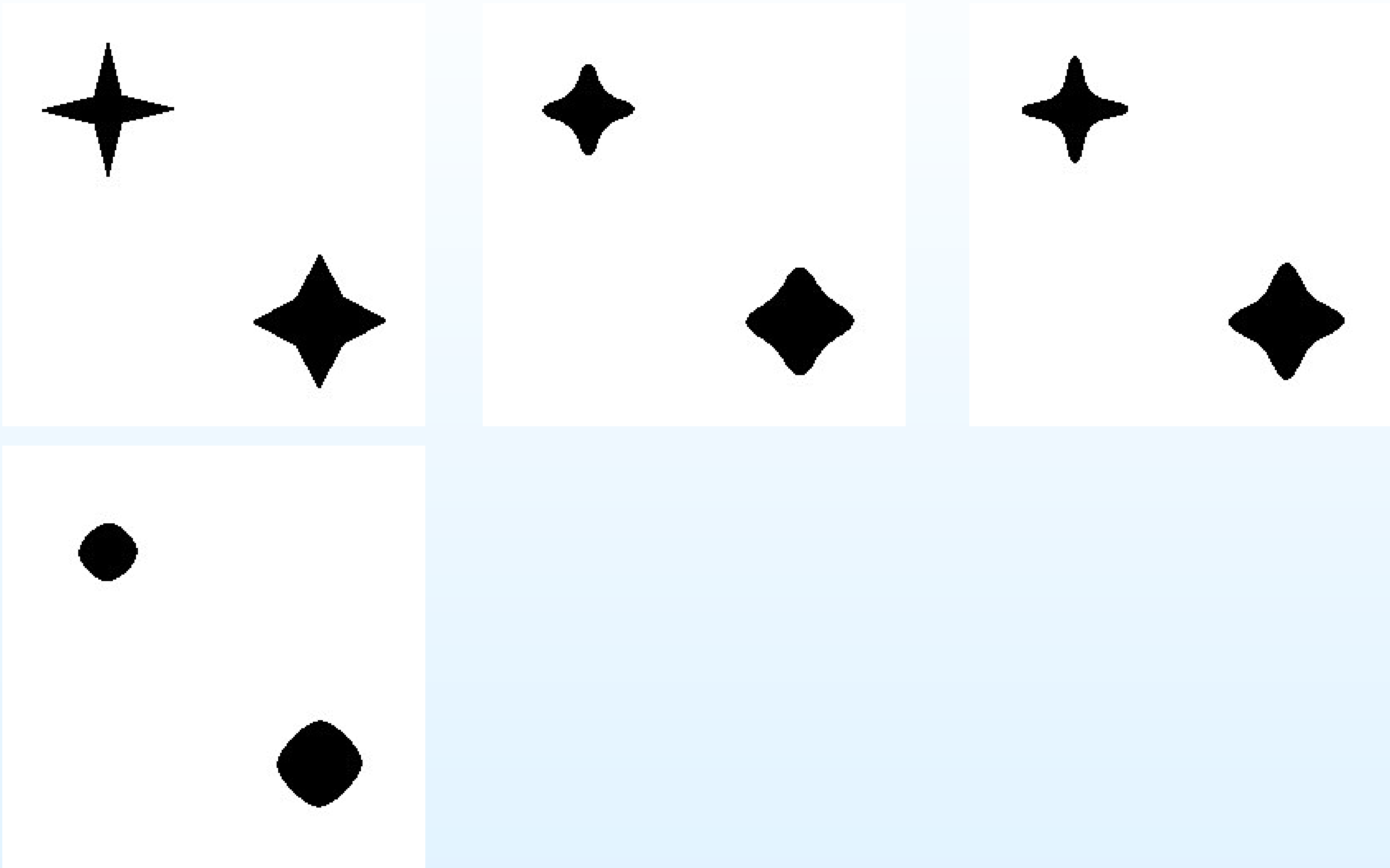




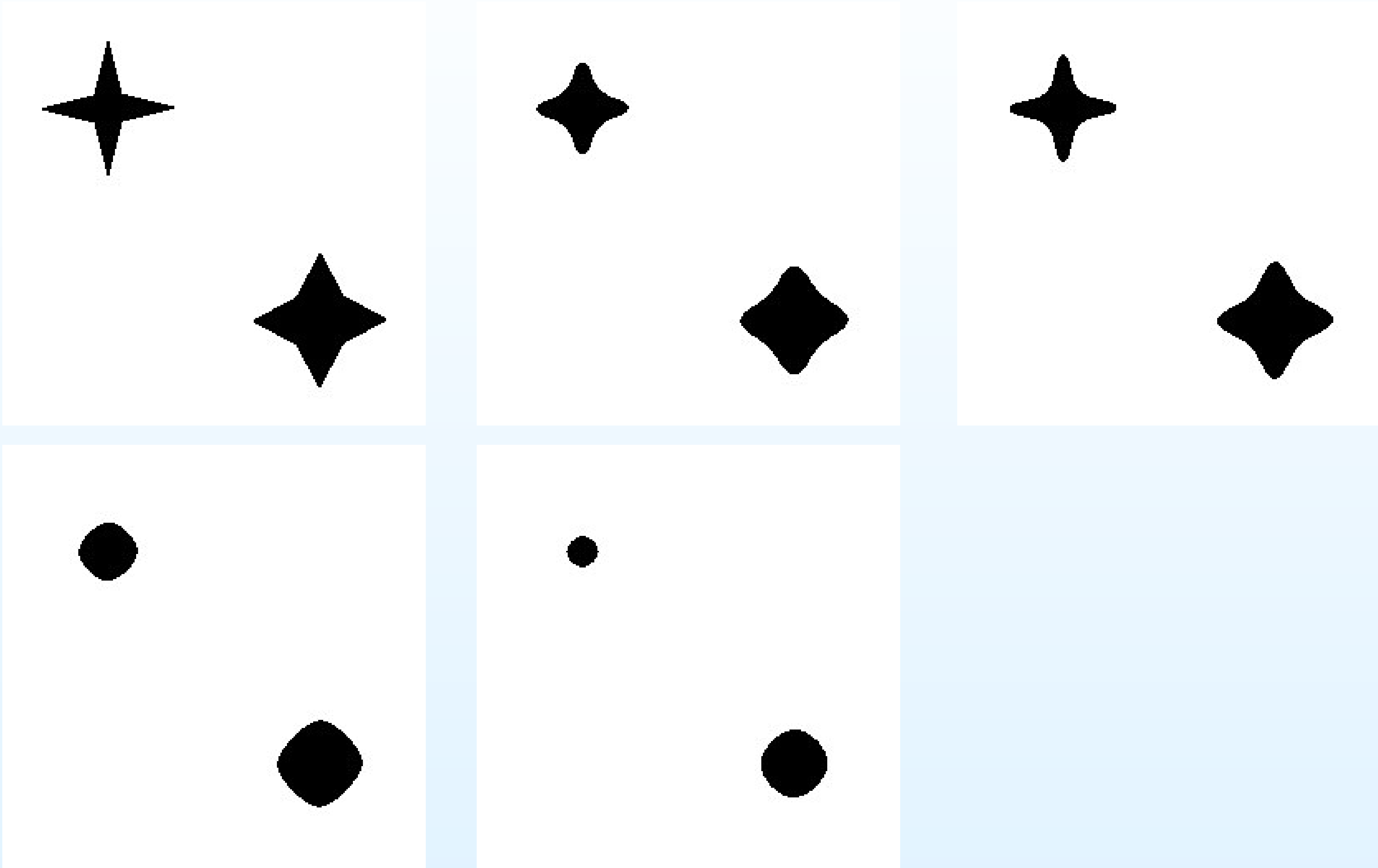
MCM scale space, for 0 , 20 , 40 iterations



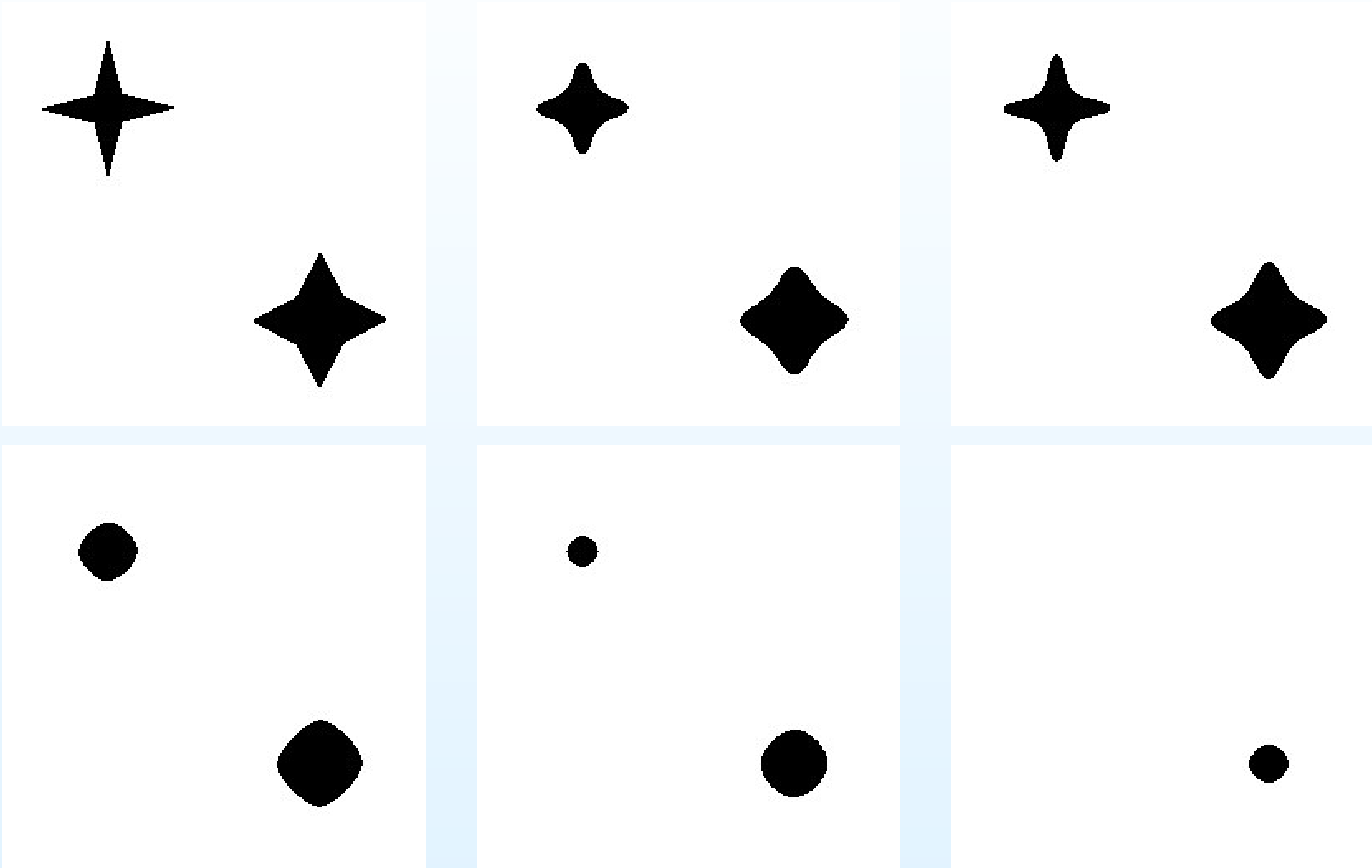
MCM scale space, for 0 , 20 , 40 , 140 iterations



MCM scale space, for 0 , 20 , 40 , 140 , 300 iterations



MCM scale space, for 0 , 20 , 40 , 140 , 300 , 500 iterations



Geometric scheme I

Idea: Decompose an image into lower level sets

$$\chi_\lambda(u) = \{x \in \mathbb{R}^2;\ u(x) \leq \lambda\}$$

or upper level sets

$$\chi^\mu(u) = \{x \in \mathbb{R}^2;\ u(x) \geq \mu\}\ .$$

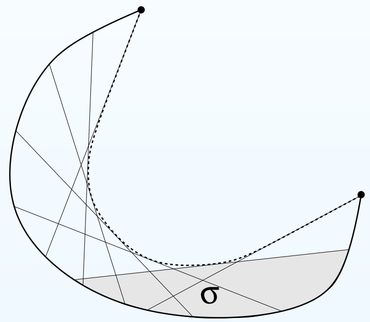
Each collection of planar sets is equivalent to the function u itself since one has the reconstruction formula

$$u(x) = \sup\{\lambda;\ x \in \chi_\lambda(u)\} = \inf\{\mu;\ x \in \chi^\mu(u)\}.$$

Then evolve the level curves and reconstruct..

Geometric scheme I (cont.)

We define affine erosion of a curve C to be the envelope of cords delimiting a region of area σ:



Geometric scheme I (cont.)

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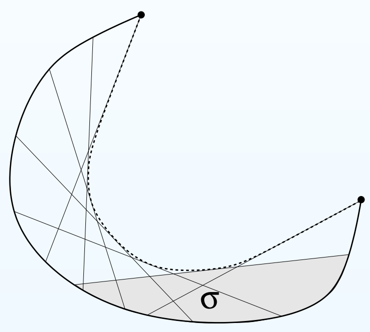
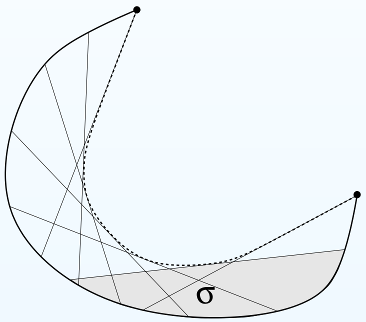


The evolution of C follows $$\frac{\partial \vec{C}}{\partial t} = \kappa^{1/3}\,\vec{N}$$

Here κ is the local curvature at a point of C and $\vec{N}$ the normal vector at this point.

Geometric scheme I (cont.)

We define affine erosion of a curve C to be the envelope of cords delimiting a region of area σ:



The evolution of C follows $$\frac{\partial \vec{C}}{\partial t} = \kappa^{1/3}\,\vec{N}$$

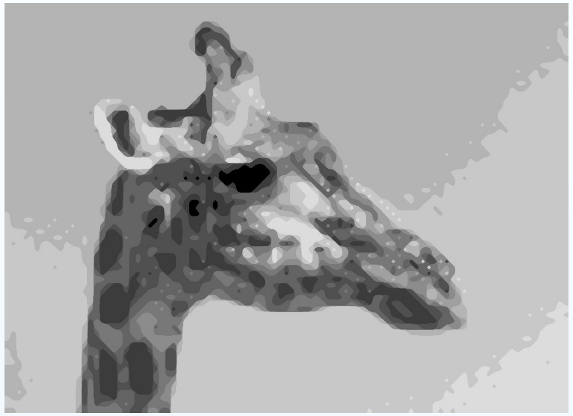
Here κ is the local curvature at a point of C and $\vec{N}$ the normal vector at this point.
Now this is the restriction to the level curves of

$$u_t = |\nabla u|\, curv(u)^{1/3}$$

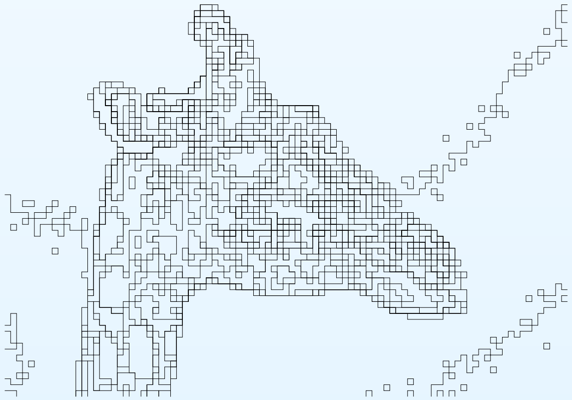
the unique morphological, affine invariant multiscale analysis.

Evolution

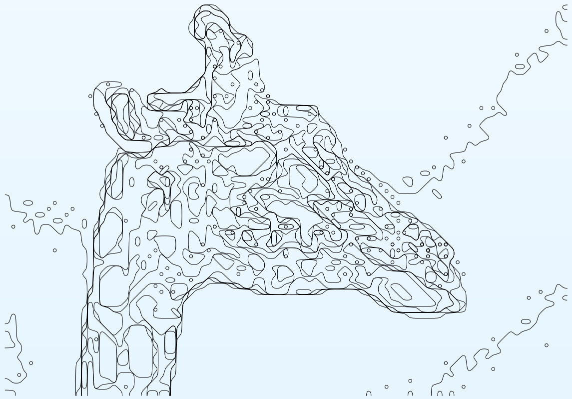
original



evolve+reconstruct



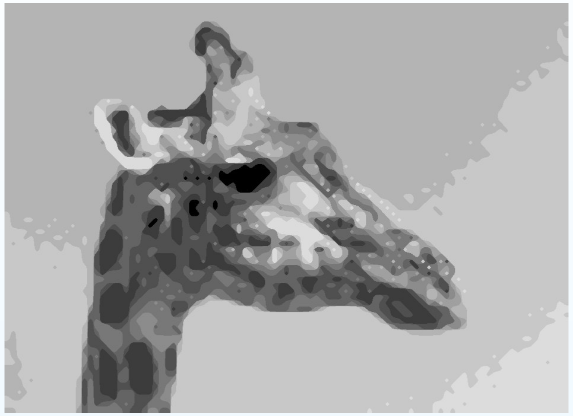
level lines



level lines

Evolution

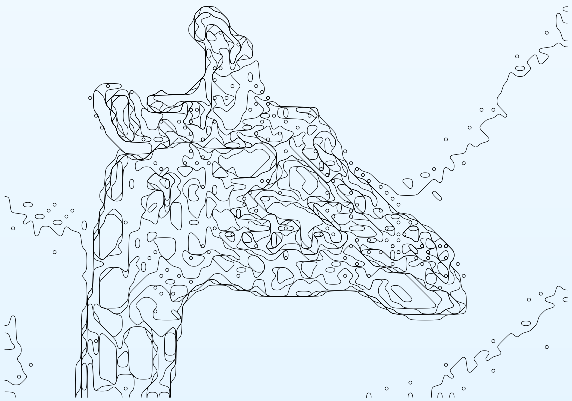
original



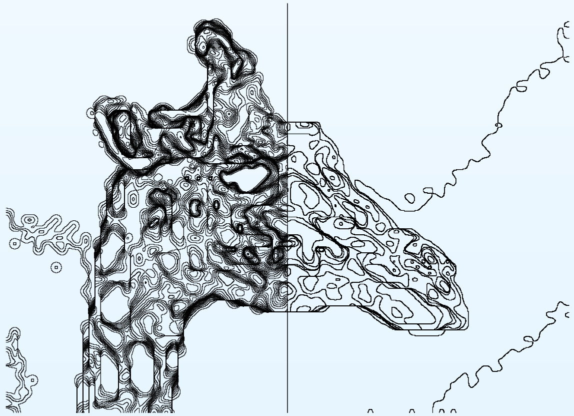
evolve+reconstruct

different schemes

level lines



level lines



scalar scheme
vs. geometric

Geometric scheme II

Idea: Integrate the total variation minimization pde

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

over the upper/lower level sets χ and with the Coarea formula:

$$\int_\chi u_t dx = \int_\chi \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) dx = \int_{\partial\chi} \frac{\nabla u}{|\nabla u|}.\vec{N} dl = \mathcal{P}(\chi)$$

where $\mathcal{P}$ = perimeter of χ .

Thus we will only modify the level not the set
by addition or subtraction of $\Delta t\, \mathcal{P}(\chi)/|\chi|$.

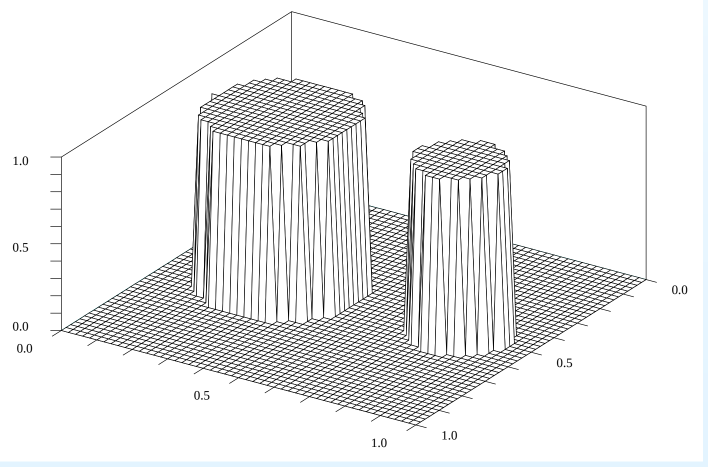
Geometric scheme II (cont.)

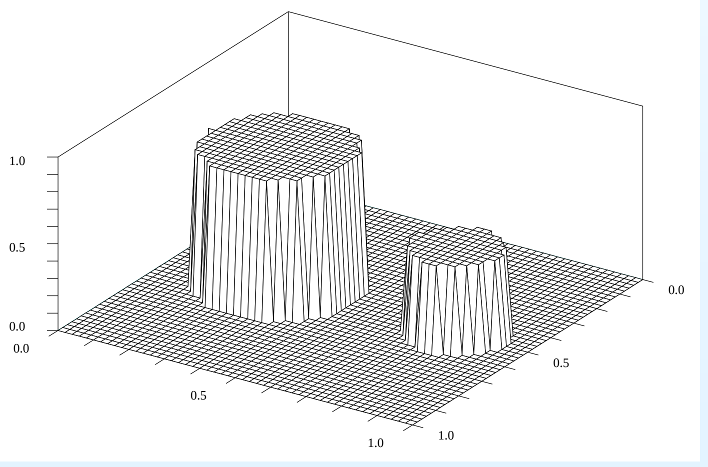
Consider discs $D_1(r_1)$ and $D_2(r_2)$ with $r_1 < r_2$.
Then $\chi^1 = \{x/u(x) \geq 1\} = D_1 \cup D_2$ and for small t:

$$\forall x \in D_1 : \quad u(t,x) = u(x) - t\,\frac{\mathcal{P}(\partial D_1)}{|D_1|} = 1 - \frac{2t}{r_1} < 1\ ,$$

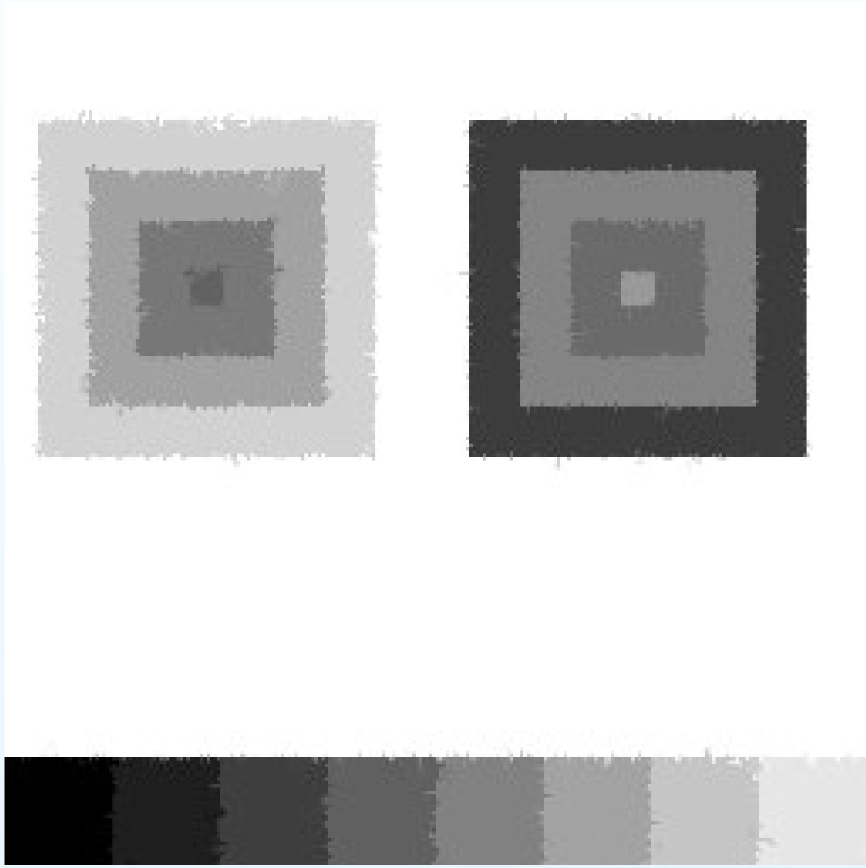
$$\forall x \in D_2 : \quad u(t,x) = u(x) - t\,\frac{\mathcal{P}(\partial D_2)}{|D_2|} = 1 - \frac{2t}{r_2} < 1\ ,$$

$$\forall x \in \Omega \backslash (D_1 \cup D_2) : \quad u(t,x) = u(x) = 0 \quad .$$

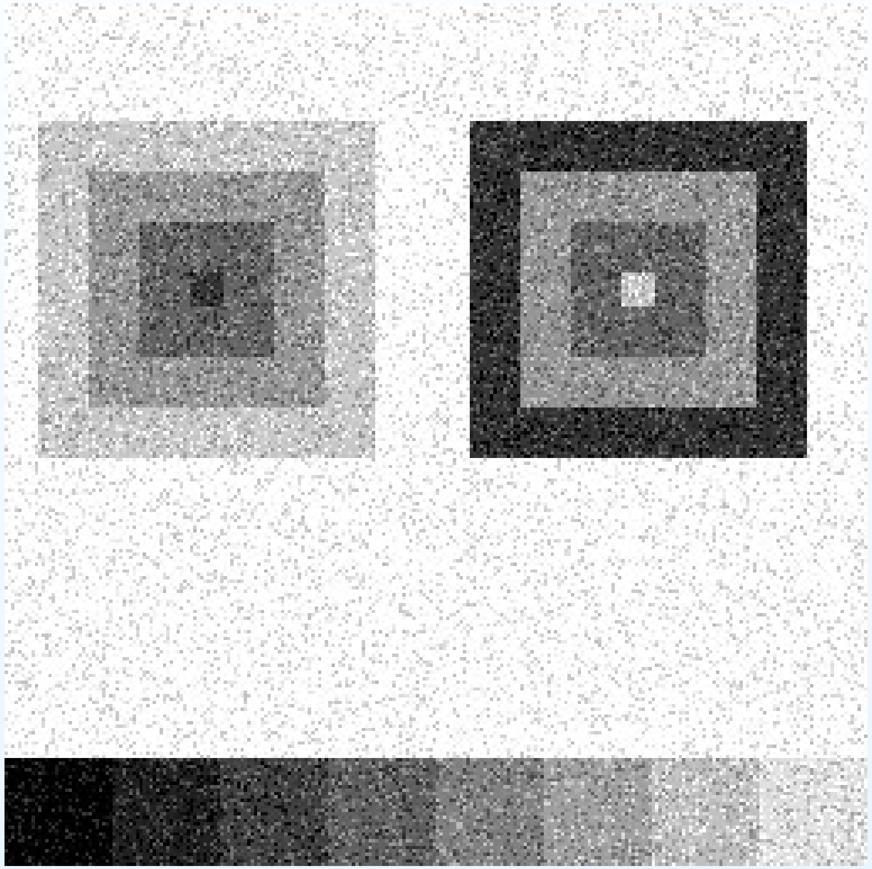




Geometric scheme II (cont.)

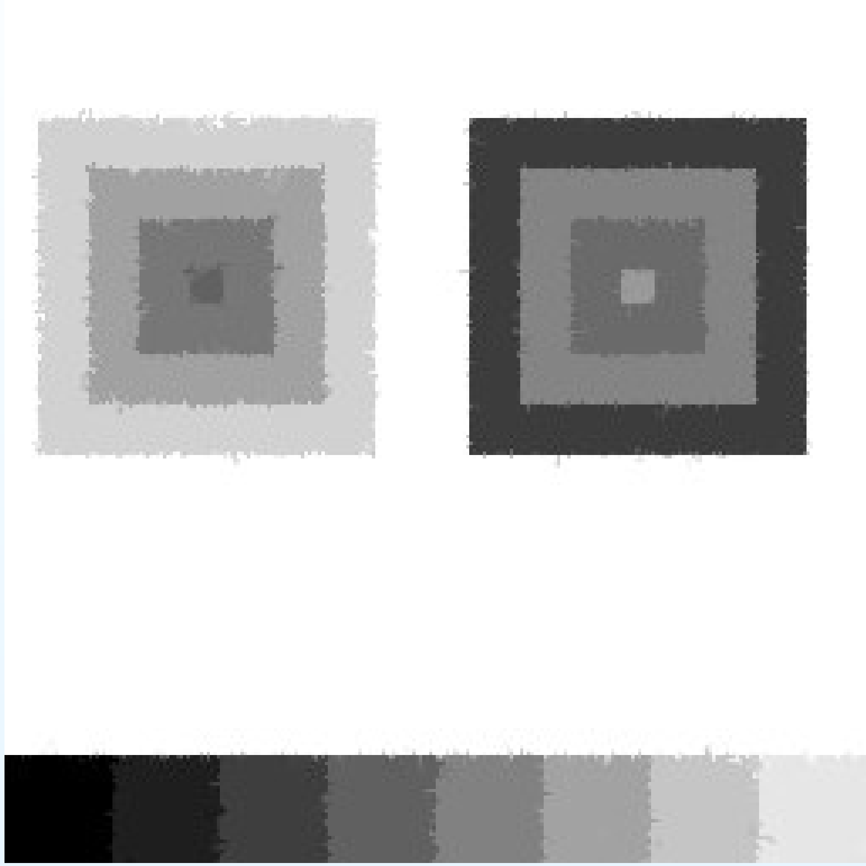


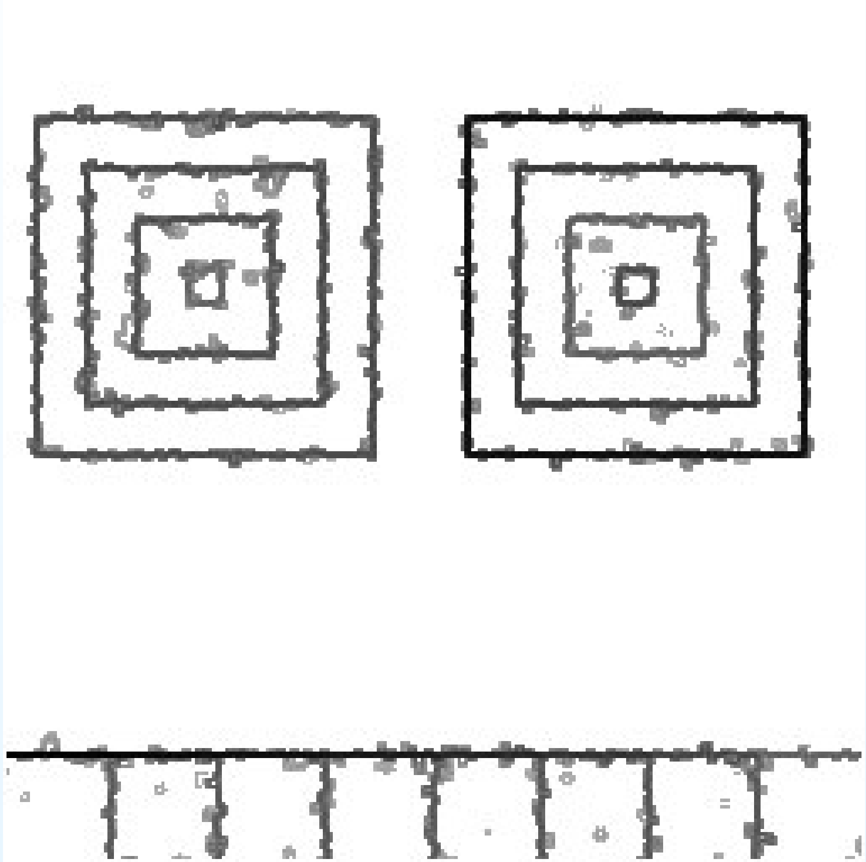
$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$



original

Geometric scheme II (cont.)





$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

$$\|\nabla u\|$$

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