We consider here a Boolean n-players version of the well-known prisoners’ dilemma. n prisoners (denoted by 1, . . . , n) are kept in separate cells. The same proposal is made to each of them: “Either you confess your accomplices (Cj, i = 1, . . . , n) or you deny them (-Cj, i = 1, . . . , n).”

Denouncing makes you freed while your partners will be sent to prison (except those who denounced you as well: these ones will be freed too).

* if none of you chooses to denounce, everyone will be freed”

Compact representation: \( G = (A, V, \pi, \Phi) \) with
- \( A = \{1, 2, \ldots, n\} \) set of players,
- \( V = \{C_1, \ldots, C_n\} \) set of propositional variables,
- \( \forall i \in \{1, \ldots, n\}, \pi_i = \{C_i\} \) control assignment function,
- \( \forall i \in \{1, \ldots, n\}, \Phi_i = \{C_i, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n\} \) utility functions.

**Pure-strategy Nash equilibria (PNE)**

A PNE is a strategy profile such that each player’s strategy is an optimum response to the other players’ strategies. \( S = \{s_1, \ldots, s_n\} \) is a pure-strategy Nash equilibrium if and only if:

\[ \forall i \in \{1, \ldots, n\}, \forall s_i' \in \{\top, \bot\}, u_i(s_i', S) \geq u_i(s_i, S) \]

The 3-players version of prisoners’ dilemma has 2 PNE: \( \{C_1, C_2, C_3\} \) and \( \{\top, \top, \top\} \).

**Characterization of PNE:**

\( S \) is a PNE for \( G \) if and only if: \( S = \bigwedge_i \{\phi_i \vee (\neg \exists i \in \{1, 2, \ldots, n\} \psi_i)\} \)

**Complexity:** Deciding whether there is a PNE in a Boolean game is \( \Sigma^P_2 \) complete.

**Introduction of preferences.**

Let \( Pref \alpha = \{\alpha_1, \ldots, \alpha_n\} \) a collection of preference relations.

- \( \alpha \) is a weak PNE (WPNE) for \( G \) if \( \forall i \in \{1, \ldots, n\}, \forall \alpha_i' \in \{\top, \bot\}, u_i(\alpha_i', S) \leq u_i(\alpha_i, S) \) and \( \exists \alpha_i \in \{\top, \bot\} \) such that \( u_i(\alpha_i', S) < u_i(\alpha, S) \).

- \( \alpha \) is a strong PNE (SPNE) for \( G \) if \( \forall i \in \{1, \ldots, n\}, \forall \alpha_i' \in \{\top, \bot\}, u_i(\alpha_i', S) < u_i(\alpha_i, S) \).

**Characterization of dominated strategies:**

- \( s_i \) is strictly dominates strategy \( s'_i \) if and only if: for all \( s' \in 2^X \), \( u_i(s_i, s') > u_i(s'_i, s') \).

- \( s_i \) weakly dominates strategy \( s'_i \) if and only if: \( u_i(s_i, s') \geq u_i(s'_i, s') \).

**Complexity:** Deciding whether a given strategy \( s'_i \) is weakly dominated is \( \Sigma^P_2 \) complete.

**2 cases.**

- **A prioritized goal base \( \Sigma \) is a collection \( \{\Sigma_1, \ldots, \Sigma_n\} \) of sets of propositional formulas.**
  - \( \Sigma^p \): set of goals of priority \( j \).
  - the smallest \( j \), the more priority the formula in \( \Sigma \).

  **Discrimin preference relation** \( S \in 2^{\Sigma^p} \) iff \( \exists k \in \{1, \ldots, p\} \) such that: \( S \cap \Sigma_k \neq \emptyset \), \( S \cap \Sigma_j = \emptyset \) and \( \forall j < k \), \( S \cap \Sigma_j = \emptyset \).

  **Leximin preference relation** \( S \in 2^{\Sigma^p} \) iff \( \exists k \in \{1, \ldots, p\} \) such that: \( S \cap \Sigma_k \neq \emptyset \), \( S \cap \Sigma_j = \emptyset \) and \( \forall j < k \), \( S \cap \Sigma_j = \emptyset \).

  **Best-out preference relation.** Let \( \alpha(s) = \min \{j \in \{1, \ldots, n\} \mid \exists \psi_i \in \Sigma, S \models \psi_i \wedge \neg \phi_i\} \) with the convention \( \alpha(s) = -1 \). Then \( S \models \exists \psi_i \models \alpha(S) \models \neg \phi_i\).

- A PG-Boolean game is a 4-sple game \( (A, V, \pi, \Phi) \), where \( \Phi = (\Sigma_1, \ldots, \Sigma_n) \).

  **NE\textsubscript{WPNE}(G) \subseteq NE\textsubscript{SPNE}(G) \subseteq NE\textsubscript{PNE}(G).**

  \( NE\textsubscript{PNE}(G) \subseteq NE\textsubscript{SPNE}(G) \subseteq NE\textsubscript{PNE}(G) \).

  \( G^{\Phi \Phi} = (A, V, \pi, \Phi^{\Phi \Phi}) \) denotes the k-reduced game of \( G \) in which all players’ goals in \( G \) are reduced in their k first stricts. \( \Phi^{\Phi \Phi} = \{\Sigma_1, \ldots, \Sigma_k\} \).

  Let \( \epsilon \in \{\text{disc, lex, bo}\} \). If \( S \) is a SPNE (resp. WPNE) for \( Pref_{\Phi^{\Phi \Phi}} = \alpha \), then \( S \) is a SPNE (resp. WPNE) for \( Pref_{\Phi^{\Phi \Phi}} = \gamma \).

  Let \( G = (A, V, \pi, \Phi) \) with \( A = \{1, 2\}, V = \{a, b, c\}, \pi_1 = \{a, c\}, \pi_2 = \{\emptyset\}, \Sigma_1 = \{\neg \Phi(\emptyset, \emptyset)\}, \Sigma_2 = \{\neg \Phi(\emptyset, \emptyset)\} \).

  **Examples of CPNets.**

  \( \pi_1 = \langle q_1, \tau_1 \rangle \) is a CP-net on \( V \), where \( q_1 \) is a directed graph over \( V \) and \( \tau_1 \) is a set of conditional preference tables \( CPF(X_i) \) for each \( X_i \in V \).

  Each \( CPF(X_i) \) associates a total order \( \succ_i \) with each instantiation \( p \in 2^n \).

  Each \( \phi \) is a CP-net on \( V \).

  Let \( G = (A, V, \pi, \Phi) \) be a CPN in which the graphs \( \phi_i \) are all identical (\( V, \pi \), \( \Phi \)) and acyclic. Then \( \gamma \) has one and only one strong PNE.

  For each player \( i, \phi_i \) is denoted by \( (V, \pi_i, \Phi_i) \) with \( \oplus_i \) being the set of edges of \( \phi_i \).

  The union graph of \( G \) is defined by \( G \uplus \Phi \equiv \{G, V, \pi, \Phi\} \), is the game obtained from \( G \) by rewriting, where:

  the graph of each player’s CP-net has been replaced by the graph of the union of CPNets of \( G \) and the CPT of each player’s CP-net are modified in order to fit with the new graph, keeping the same preferences.

  Let \( G = (A, V, \pi, \Phi) \) be a CPN. If the union graph of \( G \) is acyclic then \( G \) has one and only one SPNE.

  **Using these partial pre-orders, Nash equilibria are:** \( NE\textsubscript{WPNE} \subseteq NE\textsubscript{SPNE} \subseteq \{abc\} \).

  It is possible to verify then the union graph is acyclic.